



CK-12 Texas Instruments Calculus

Teacher's Edition



CK-12 Texas Instruments Calculus Teacher's Edition

Say Thanks to the Authors

Click <http://www.ck12.org/saythanks>

(No sign in required)



To access a customizable version of this book, as well as other interactive content, visit www.ck12.org

EDITOR

Lori Jordan

CK-12 Foundation is a non-profit organization with a mission to reduce the cost of textbook materials for the K-12 market both in the U.S. and worldwide. Using an open-content, web-based collaborative model termed the **FlexBook®**, CK-12 intends to pioneer the generation and distribution of high-quality educational content that will serve both as core text as well as provide an adaptive environment for learning, powered through the **FlexBook Platform®**.

Copyright © 2014 CK-12 Foundation, www.ck12.org

The names “CK-12” and “CK12” and associated logos and the terms “**FlexBook®**” and “**FlexBook Platform®**” (collectively “CK-12 Marks”) are trademarks and service marks of CK-12 Foundation and are protected by federal, state, and international laws.

Any form of reproduction of this book in any format or medium, in whole or in sections must include the referral attribution link <http://www.ck12.org/saythanks> (placed in a visible location) in addition to the following terms.

Except as otherwise noted, all CK-12 Content (including CK-12 Curriculum Material) is made available to Users in accordance with the Creative Commons Attribution-Non-Commercial 3.0 Unported (CC BY-NC 3.0) License (<http://creativecommons.org/licenses/by-nc/3.0/>), as amended and updated by Creative Commons from time to time (the “CC License”), which is incorporated herein by this reference.

Complete terms can be found at <http://www.ck12.org/terms>.

Printed: November 9, 2014

flexbook
next generation textbooks



Contents

1	TE Introduction to Calculus - TI	1
1.1	Calculus TI Resources	2
2	TE Functions, Limits, and Continuity - TI	3
2.1	Modeling Data	4
2.2	What is Calculus?	6
2.3	One Sided Limits	8
3	TE Differentiation - TI	11
3.1	Gettin' the Swing	12
3.2	Move Those Chains	15
3.3	Implicit Differentiation	17
4	TE Applications of Derivatives - TI	23
4.1	Xtreme Calculus	24
4.2	Helicopter Bungee Jump	28
4.3	Optimization	31
4.4	Linear Approximation	33
5	TE Integration - TI	36
5.1	Exploring the Area Under a Curve	37
5.2	Sum Rectangles	40
5.3	FTC Changed History	43
6	TE Applications of Integration	46
6.1	The Area Between	47
6.2	Volume by Cross Sections	49
6.3	Gateway Arc Length	51
7	TE Transcendental Functions - TI	54
7.1	Inverses of Functions	55
7.2	The Logarithmic Derivative	58
8	TE Integration Techniques - TI	60
8.1	Integration by Substitution	61
8.2	Integration by Parts	64
8.3	Charged Up	69
9	TE Infinite Series - TI	72
9.1	Exploring Geometric Sequences	73
9.2	Infinite Geometric Series	76
9.3	Mr. Taylor, I Presume?	79
9.4	AP* Calculus Exam Prep	83

CHAPTER

1

TE Introduction to Calculus - TI

Chapter Outline

1.1 **CALCULUS TI RESOURCES**

1.1 Calculus TI Resources

Teacher's Edition

Introduction

This FlexBook® resource contains Texas Instruments (TI) Resources for the TI-89, unless otherwise indicated in the description of the activity. All the activities in this resource supplement the lessons in the student edition. All activities are listed in the same order as the Student Edition. You may find it helpful to copy all or part of the included resource pages for the students in order for the activities to go more smoothly.

There are also corresponding links in the student edition of the CK-12 Calculus FlexBook® textbook, <http://www.ck12.org/book/CK-12-Calculus/>.

Any activity that requires a calculator file or application, go to <http://www.education.ti.com/calculators/downloads> and type the name of the activity or program in the search box.

CHAPTER

2**TE Functions, Limits, and
Continuity - TI****Chapter Outline**

- 2.1 MODELING DATA**
 - 2.2 WHAT IS CALCULUS?**
 - 2.3 ONE SIDED LIMITS**
-

2.1 Modeling Data

This activity is intended to supplement Calculus, Chapter 1, Lesson 3.

ID: 12316

Time required: 20 minutes

Activity Overview

In this activity, students will graph data modeling exponential and logarithmic growth and find equations representing the data.

Topic: Modeling Growth

- Exponential Growth and Decay
- Compound Interest
- Logarithmic Growth

Teacher Preparation and Notes

- This activity can be used with the TI-83, 84, or 89. This supplement is geared towards the 83 and 84.
- Students should be aware of the compound interest formula in order to be able to find the growth of the exponential data given in Problem 1.
- Students should be aware of basic logarithms in order to determine the base of the log function for Problem 2.
- The lists L1 to L6 will need to be loaded into the student calculators before beginning this activity.
- To download the list files, go to <http://www.education.ti.com/calculators/downloads/US/Activities/Detail?id=12316> and select L1-L6.

Associated Materials

- Student Worksheet: Modeling Data <http://www.ck12.org/flexr/chapter/9726>
- L1.8x1, L2.8x1, L3.8x1, L4.8x1, L5.8x1, and L6.8x1

Problem 1 –Exponential Growth

In this problem, students will graph data given in L1 and L2.

Students should determine an equation for the data using their knowledge of compound interest equations.

The equation is $y = 1000(1.04)^x$.

Students can plot the function to determine if it goes through the graphed points by pressing $Y =$ and entering the function.

Discussion Questions

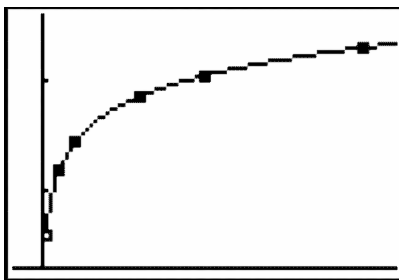
- What variable should be on the horizontal axis? Vertical axis?
- How can you determine the interest rate for this growth?

Problem 2 –Logarithmic Growth

In this problem, students will graph data given on $L1$ and $L2$. Students should determine an equation for the data using the regression capabilities of the graphing calculator.

Note: to find $Y1$, press **VARs**, scroll over to **Y-VARS**, select **1:Function...**, and select **1 : $Y1$** .

The equation is $y = 8.6 \times 10^{-8} + 0.369 \times \ln x$.



Discussion Questions

- What variable should be on the horizontal axis? Vertical axis?

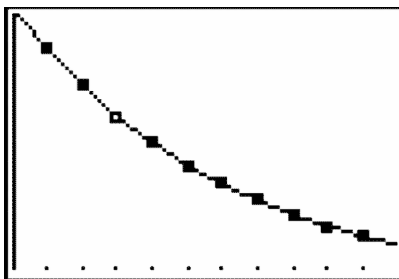
Extension Problem –Exponential Decay

In this problem, students will continue with the process in the first two problems. The given data represents exponential decay.

The equation is $y = 1250(0.85)^x$.

Discussion Questions

- What is the number of acres the farmer started with in year zero?
- By what percent does the amount of acres available decrease every year?



2.2 What is Calculus?

This activity is designed to supplement Calculus, Chapter 1, Lesson 4.

ID: 12726

Time Required: 20 minutes

Activity Overview

Students will explore the major concepts of calculus. Students will answer questions and investigate animations in this introduction and overview of calculus. Students will get the big picture of what calculus is about and its various applications.

Topic: Introduction and Overview of Calculus

- *Introduce concepts of limit, derivative and integral.*
- *Brief historical background of calculus and its application.*

Teacher Preparation and Notes

- *This activity is designed to be a student-centered activity to begin the year of calculus. It can also be used toward the end of the year in precalculus as a preview for what is to come next year. Teachers could also use the animations with their ViewScreen™ to introduce the key concepts of calculus.*
- *Use the TI-Connect or TI-Navigator software to send the group file to students' graphing calculators. By using the calculator-to-calculator connecting cable, the files can also be quickly and easily distributed to the entire class.*
- *To download the calculator files, go to <http://www.education.ti.com/calculators/downloads/US/Activities/Detail?id=12726> and select the .89i and .89p, files (17 total).*

Associated Materials

- Student Worksheet: What is Calculus? <http://www.ck12.org/flexr/chapter/9726> , scroll down to second activity.
- main.gr1, main.gr2, main.gr3, main.gr4, main.gr5, main.gr6 (89i files)
- main.graph1, main.graph2, main.graph3, main.graph4, main.graph5 (89i files)
- main.pic1, main.pic2, main.pic3, main.pic4, main.pic5 (89i files)
- main.whatcalc.89p

Part 1 –Introduction & Limits

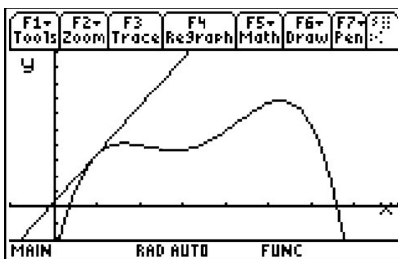
Students will read the information given on their worksheet to become introduced to the major concepts of calculus. This information will then be used to answer questions on the student worksheet. The activity begins with an overview and then transitions to limits.

Solutions

1. Students may have heard a variety of things about calculus, some true and others not so much. This question can be helpful to identify and correct misconceptions.
2. The value of $f(x)$ appears to be getting close to the value marked L . L is the limit of $f(x)$ as x approaches c .

Part 2 –Derivatives & Optimization

Part 2 transitions to the concept of the derivative. Important calculus terminology is used, including extrema, maxima, minima, neighborhood, and local max and min. Students can use the animation of a 4th degree polynomial to answer questions involving extrema, slope, and local maximum and minimum points.



After viewing each graph using the program *whatcalc*, students do not need to exit the program. There are no calculations that need to do outside of the program, so they can just select the next menu item when they are ready for that portion of the activity. After students are finished with each graph, they should press **ENTER** to return to the menu. If students do select **5:Quit** from the menu, they will need to start the program again from the Home screen.

Solutions

3. This 4th degree polynomial has 3 extrema.
4. The slope of the tangent at a max or min is zero.
5. a. The approximate maximum volume is 62.22 in^3 when the height h is 1.9 in.
- b. With calculus, the exact maximum can be found by finding the derivative and setting it equal to zero.

Part 3 –Integrals & History

Finally the area under the curve is introduced and applications to the Fundamental Theorem of Calculus are discussed.

Solutions

6. The rectangle has an area of 24.
7. The volume is found by rotating the function about $y = 0$, or the x -axis.
8. Newton and Leibniz independently co-discovered calculus.
9. Sample answer: Limit is what the function is close to when the x -value gets close to c . The derivative is the slope of the tangent and has to do with rate of change. The integral is the sum of infinitesimals and is related to the area of under the curve.

2.3 One Sided Limits

This activity is intended to supplement Calculus, Chapter 1, Lesson 6.

ID: 10994

Time required: 15 minutes

Activity Overview

*Students will graph piecewise functions and evaluate both the left hand limit and the right hand limit of the function as x approaches a given number, c . The **Trace** feature will be used to graphically estimate the one sided limit. Students will also use a table of values of each function to numerically verify that the values of the function to left and right of c are approaching the same number.*

Topic: Limits

- *One Sided Limits*

Teacher Preparation and Notes

- *Students should already have been introduced to one-sided limits.*
- *Students should know that a limit exists if and only if the left hand limit and the right hand limit are equal.*
- *Upgrade the TI-89 Titanium to OS Version 3.10 so that “Discontinuity Detection” can be utilized. On a graph, press F1 >Format to turn Discontinuity Detection ON.*

Associated Materials

- Student Worksheet: One-Sided Limits <http://www.ck12.org/flexr/chapter/9726> , scroll down to the third activity.
- Discontinuity Detection

The student worksheet gives key press instructions to set up the window so that their graphs look like the following.

Problem 1

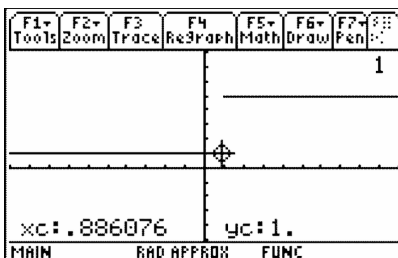
Before changing the value of a , students will graphically estimate the limit of $y_1(x)$ as x approaches 1 from the left and the right. Students will also use the table to numerically estimate the value of a that will ensure that the limit of $y_1(x)$ as x approaches one exists.

Solutions

$$\lim_{x \rightarrow 1^-} y1(x) \approx 1;$$

$$\lim_{x \rightarrow 1^+} y1(x) \approx 5;$$

$$a \approx 1$$



Problem 2

Problem 1 is repeated for a different function. Before changing the value of a , students will graphically estimate the limit of $y2(x)$ as x approaches 1 from the left and the right. Students will also use the table to numerically estimate the value of a that will ensure that the limit of $y2(x)$ as x approaches one exists. Here the algebraic calculations for the left and right hand limits are to be shown.

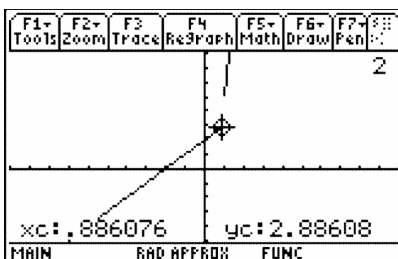
Solutions

$$\lim_{x \rightarrow 1^-} y2(x) \approx 3; \lim_{x \rightarrow 1^+} y2(x) \approx 5; a \approx 3$$

$$\lim_{x \rightarrow 1^-} y2(x) = 1 + 2 = 3$$

$$\lim_{x \rightarrow 1^+} y2(x) = a \cdot (1^2) = a$$

$$\text{So } 1 + 2 = a \cdot 1^2; \quad a = 3$$



Problem 3

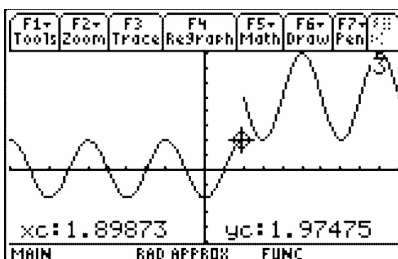
Problems 1 and 2 are repeated for a different function. Before changing the value of a , students will graphically estimate the limit of $y3(x)$ as x approaches 2 from the left and the right. Students will also use the table to numerically estimate the value of a that will ensure that the limit of $y3(x)$ as x approaches two exists. Here the algebraic calculations for the left and right hand limits are to be shown.

Students should view the table near $x = 2$ instead of 1.

Solutions

$$\lim_{x \rightarrow 2^-} f(x) \approx 2; \lim_{x \rightarrow 2^+} f(x) \approx 5; a \approx 2;$$

$$2 \sin\left(\frac{\pi}{2}(2-1)\right) = 3 \sin\left(\frac{\pi}{2}(2-4)\right) + a; a = 2$$

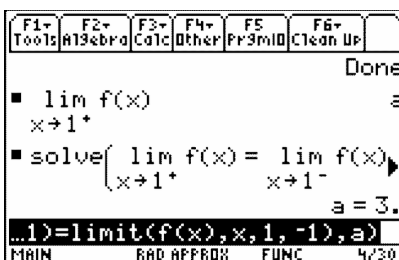


Extension –Continuity

Students are introduced to the concept of continuity and are asked to prove the functions in Problems 2 and 3 are continuous. They are also instructed how to use CAS to algebraically solve for a that makes the limit exist.

Solutions

Students must show that all three conditions are met in order to satisfy the criteria for continuity.



CHAPTER

3

TE Differentiation - TI

Chapter Outline

- 3.1 GETTIN' THE SWING
 - 3.2 MOVE THOSE CHAINS
 - 3.3 IMPLICIT DIFFERENTIATION
-

3.1 Gettin' the Swing

This activity is intended to supplement Calculus, Chapter 2, Lesson 4.

ID: 11691

Time Required: 40-45 minutes

Activity Overview

In this activity, students will investigate sinusoidal functions, collect data from a swinging pendulum, model the data with a cosine function, and take the derivative to find the velocity and acceleration.

Topic: Model Sinusoidal Data

- Write an equation for a sinusoidal function.
- First and second derivative to determine velocity and acceleration

Teacher Preparation and Notes

- This activity utilizes a CBR or CBR 2, which records the movement of the pendulum and transmits it to the TI-89.
- Students will need to know that the derivative of position is velocity, and the derivative of velocity is acceleration. Students are also expected to know the derivative of trigonometric functions.
- The activity is designed to be a student-centered discovery/review involving data collection. **If a motion sensor is not available, then use the data from the lists provided.**
- In order to collect data using the Calculator-Based Ranger 2TM, use the Ranger program. The Ranger program should be in the CBR device. To download it onto your calculator, connect the two, press 2ndLINK, arrow over to RECEIVE, and press ENTER.
- To download the calculator files, go to <http://www.education.ti.com/calculators/downloads/US/Activities/Detail?id=11691> and select time, distance, and velocity.

Associated Materials

- Student Worksheet: Getting the Swing <http://www.ck12.org/flexr/chapter/9727>
- time.89l, distance.89l, velocity.89l
- Calculator-Based Ranger (CBR) 2TM or Vernier GO! TM Motion
- pendulum (a ball hanging from a string)
- meter stick
- stop watch

Part 1 –Warm-up

Students begin the activity by answering questions to help them visualize and review characteristics of sinusoidal functions.

Next, students will translate the cosine function to explore the effects of varying the parameters of the general sinusoidal functions. Students will use a split screen to view the equation and graph of the function at the same time. Students should see that as B increases the period decreases. Students should also see that this relationship involves 2π .

Further Discussion

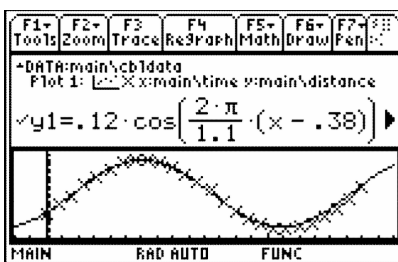
- Ask students what the derivative of $y = \cos(x)$ is. When the $A = 1$, $B = 1$, $C = 0$, and the D is set at $\frac{\pi}{2}$, the graph looks like $y = -\sin(x)$.

Solutions

1. range is from -1 to 1
2. amplitude $A = 1$
3. period $T = 2\pi$
4. A makes the amplitude bigger
5. C produces a vertical shift
6. B changes the frequency
7. $B = \frac{2\pi}{T}$; The inverse function graphed on 3.2 confirms this.
8. $y = A \cdot \cos(B \cdot (x - D)) + C$

Part 2 –Collect & Analyze Data

This part of the activity begins with students plugging in the CBR 2™ or GO!™ Motion with an I/O cable. (Note: the mini USB cables will not work with this program.) Students should start the *Ranger* program. The experiment should be set up to match the settings shown on the student worksheet. When **ENTER** is pressed, data is collected. If there is a wall, chair, person or some other object that “distracts” the motion detector, the data may not look as good as depicted on the right. Students can repeat the experiment by pressing **ENTER** and select **3:Repeat Sample**. Have student repeat the data collection until they are pleased with the shape of the graph.



Be sure to use a smooth ball so that the sonic pulse is reflected off the surface well.

Students can see the connection to real life more clearly if meter sticks and stop watches are available. Students can measure the period with the stop watch. With the meter stick they can see how far the pendulum is from the motion detector when the ball is at rest. This will be their C . Also they can measure how far they pull the pendulum. This is the amplitude. The phase shift is related to when they pressed start to collect the data.

Solutions

The graphs and equations for the experiment will vary. The following equations correlate to the data in the given lists.

$$y = 0.12 \cos \left(\frac{2\pi}{1.1} \cdot (x - 0.38) \right) + 0.35$$

$$v = -0.12 \left(\frac{2\pi}{1.1} \right) \sin \left(\frac{2\pi}{1.1} \cdot (x - 0.38) \right)$$

$$a = -0.12 \left(\frac{2\pi}{1.1} \right)^2 \cos \left(\frac{2\pi}{1.1} \cdot (x - 0.38) \right)$$

3.2 Move Those Chains

This activity is intended to supplement Calculus, Chapter 2, Lesson 5.

ID: 11364

Time Required: 15 minutes

Activity Overview

In this activity, students will explore the Chain Rule. Students are first asked to make a conjecture about the derivative of $f(x) = (2x + 1)^2$ based on the Power Rule. They are then asked to graph their derivative function and compare it to the graph of $f'(x)$. They will then examine “true” statements about various derivatives of composite functions. They will observe patterns and use these patterns to create a rule for finding the derivatives of other composite functions. They will then use their rule to create “true” examples of their own.

Topic: Chain Rule

- Derivative of a composite function

Teacher Preparation and Notes

- Students will type $d(f(x),x)$ in the entry line of the TI-89. When they press **ENTER**, the TI-89 will return the expression in 'pretty print', and it will appear as $\frac{d}{dx}(f(x))$.
- Note: Some functions will have an independent variable other than x to familiarize students with using other variables.
- The true statements show the derivatives in unsimplified form so that students will more easily identify the patterns.

Associated Materials

- Student Worksheet: Move those Chains <http://www.ck12.org/flexr/chapter/9727> , scroll down to the second activity.

Problem 1

Students are asked to make a conjecture about what the derivative of $f(x) = (2x + 1)^2$ is using the Power Rule. From the home screen, students are to go to the $Y = \text{Editor}$ by pressing \blacklozenge [$Y =$]. Students should enter the function in $y1$, **nDeriv**($y1(x),x$) in $y2$, and their conjecture about the derivative of $f(x) = (2x + 1)^2$ in $y3$. (To access the **nDeriv** command, go to the Math menu (2^{nd} [**MATH**]) and select **B:Calculus >A:nDeriv(.)** Students should graph only functions $y2$ and $y3$. (To deselect a function, highlight the function and press $F4$. Note: The graph may take a minute to appear. If a student's conjecture is correct, the graphs of $y2$ and $y3$ will coincide. If a student's conjecture is incorrect the graphs of $y2$ and $y3$ will not coincide.

Students are asked to expand $(2x + 1)^2$, and take the derivative of this expression term by term. They are then to compare it to $y3(x)$.

Solutions

1. Sample answer: $2(2x + 1)$
2. Sample answer: No, my answer was not correct. I can try expanding the function before taking the derivative.
- 3.

$$\begin{aligned}
 \frac{d}{dx}((2x+1)^2) &= \frac{d}{dx}((2x+1)(2x+1)) \\
 &= \frac{d}{dx}(4x^2 + 4x + 1) \\
 &= \frac{d}{dx}(4x^2) + \frac{d}{dx}(4x) + \frac{d}{dx}(1) \\
 &= 8x + 4
 \end{aligned}$$

Problem 2

Students are asked to examine “true” statements of the derivatives of composite functions while looking for patterns. They are asked to discuss the patterns they observed with fellow students.

Students are asked to use the pattern observed to make “true” derivative of composite function statements. If the handheld does not return the word ‘true,’ students can try again by pressing 2nd **ENTER**, editing their solution, and pressing **ENTER** again.

The Chain Rule is presented to students, and they are asked to write three additional true statements.

Solutions

4. Sample answer: The Power Rule is applied to the “outer” function, then is multiplied by the derivative of the “inner” function.
5. $\frac{d}{dx}((3x+2)^2) = 2 \cdot (3x+2)^1 \cdot 3$
6. $\frac{d}{dx}((7x+2)^3) = 3 \cdot (7x+2)^2 \cdot 7$
7. $\frac{d}{dx}((5x^2+2x+3)^4) = 4 \cdot (5x^2+2x+3)^3 \cdot (10x+2)$
8. Answers may vary.

Problem 3 –Homework Problems

Students are given five additional exercises that can be used as homework problems or as extra practice during class.

Solutions

1. $\frac{d}{dx}((4x^3+1)^2) = 2 \cdot (4x^3+1)^1 \cdot (12x^2)$
2. $\frac{d}{dx}((-5x+10)^7) = 7 \cdot (-5x+10)^6 \cdot (-5)$
3. $\frac{d}{dx}((2t^5-4t^3+2t-1)^2) = 2 \cdot (2t^5-4t^3+2t-1)^1 \cdot (10t^4-12t^2+2)$
4. $\frac{d}{dx}((x^2+5)^{-2}) = -2 \cdot (x^2+5)^{-3} \cdot 2x$
5. $\frac{d}{dz}((z^3-3z^2+4)^{-3}) = -3 \cdot (z^3-3z^2+4)^{-4} (3z^2-6z)$

3.3 Implicit Differentiation

This activity is intended to supplement Calculus, Chapter 2, Lesson 6.

ID: 8969

Time required: 45 minutes

Activity Overview

Students will be introduced to the concept of implicit differentiation. Students will start by solving a relation for y and using methods for which they are already familiar to find the derivative of this relation. Students will then use the **impDif** command to find an alternate form of the relation. Students will verify that these two forms are equal. Students will also learn how to perform implicit differentiation by hand, using the **impDif** command to check their results. Students will be asked to find the numerical derivative of a relation for specific x -values. A graphical connection to the results found using the implicit differentiation will also be made.

Topic: Formal Differentiation

- Use implicit differentiation to find the derivative of a function defined implicitly.
- Use the command **impDif** to check the manual computation of implicit differentiation.

Teacher Preparation and Notes

This investigation offers an opportunity to introduce the concept of implicit differentiation.

- Students should be familiar with finding derivatives of functions where y is explicitly defined in terms of x . Students should also be familiar with the Chain Rule.
- This activity is designed to be **student-centered** with the teacher acting as a facilitator while students work cooperatively. The student worksheet is intended to guide students through the main ideas of the activity and provide a place for them to record their observations.
- Reference the Chain Rule to emphasize that y' (or $\frac{dy}{dx}$) must accompany taking the derivative of expressions containing y
- Provide students with additional practice finding derivatives using implicit differentiation. Include examples such as $\sin(2x - 7xy) = 16y$ that require using the chain rule, product rule, etc ...
- Before starting this activity, students should go to the home screen and select **F6 : Clean Up >2: NewProb**, then press **ENTER**. This will clear any stored variables, turn off any functions and plots, and clear the drawing and home screens.

Associated Materials

- Student Worksheet: Implicit Differentiation <http://www.ck12.org/flexr/chapter/9727> , scroll down to the third activity.

One focus question defines this activity: *How can you find the derivative of a relation, $F(x,y)$ that is not solved for y ?*

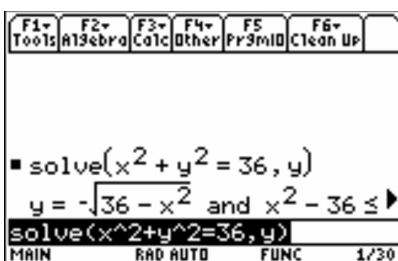
Use the circle $x^2 + y^2 = 36$ as a discussion point for this focus question. Encourage students to realize that $x^2 + y^2 = 36$ can be solved for y , giving two equations—one that defines the top semicircle and one that defines the bottom semicircle. Have them graph both equations on the same set of axes.

Explain to the students that they will in fact solve $x^2 + y^2 = 36$ for y and use the resulting equations to find the derivative of $x^2 + y^2 = 36$.

Problem 1 –Finding the derivative of

Step 1: Students will solve $x^2 + y^2 = 36$, for y . They should be able to this by hand. However, if they use the **Solve** command, **F2 :Algebra >1:Solve**, they will get the two functions with the restrictions on the variables:

$$f_1(x) = \sqrt{36 - x^2} \text{ and } f_2(x) = -\sqrt{36 - x^2}.$$



Substituting these equations in the original relation yields $x^2 + \left(\sqrt{36 - x^2}\right)^2 = 36$ and $x^2 + \left(-\sqrt{36 - x^2}\right)^2 = 36$. These equations can be simplified as follows:

$$\begin{array}{lcl} x^2 + \left(\sqrt{36 - x^2}\right)^2 = 36 & \text{and} & x^2 + \left(-\sqrt{36 - x^2}\right)^2 = 36 \\ x^2 + (36 - x^2) = 36 & & x^2 + (36 - x^2) = 36 \\ 36 = 36 & & 36 = 36 \end{array}$$

Now students are to graph both functions on the same set of axes and get a circle. Due to the scale of the graphing screen, the circle may be distorted. Students can select **F2 :Zoom >5:ZoomSqr** to adjust the scale. Discuss what problems they might have if they want to find the derivative at $x = 2$.

Step 2: The derivatives of $f_1(x)$ and $f_2(x)$ are: $\frac{d}{dx}f_1(x) = \frac{-x}{\sqrt{36 - x^2}}$ and $\frac{d}{dx}f_2(x) = \frac{x}{\sqrt{36 - x^2}}$.

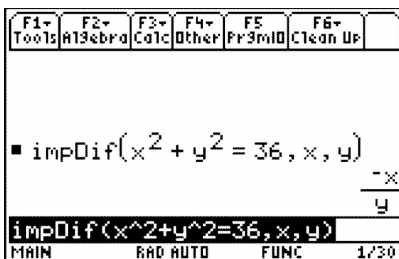
The students should be able to do this by hand. If they choose to use the TI-89 Titanium graphing calculator they will get the screen at the right. The keystrokes are **F3 :calc >1:d(differentiate** or press 2^{nd} [d].

Using these derivatives, the slopes at $x = 2$ are found to be:

$$\frac{-2}{\sqrt{36 - 2^2}} = -\frac{\sqrt{2}}{4} \approx -0.354 \text{ and } \frac{-(-2)}{\sqrt{36 - (-2)^2}} = \frac{\sqrt{2}}{4} \approx 0.354.$$

Students should be encouraged to look at the graph to see if these slopes make sense. They will see that the slopes for any value of x should be equal in magnitude and opposites for each other.

Step 3: Students will use the **impDif** command, shown at the right, to find an alternate form for the derivative of $x^2 + y^2 = 36$. To access the command press **F3 :calc >D:impDif**(, where the independent variable is listed first and the dependent variable is listed second.



Students will use $\frac{dy}{dx} = \frac{-x}{y}$ to find the slope of the tangents to $x^2 + y^2 = 36$. First they will need to solve for y , knowing that $x = 2$: $(2)^2 + y^2 = 36 \rightarrow y^2 = 32 \rightarrow y = \pm 4\sqrt{2}$

Substituting $(2, 4\sqrt{2})$ and $(2, -4\sqrt{2})$ into the formula for the derivative yields

$$\frac{dy}{dx} = \frac{-2}{4\sqrt{2}} = -\frac{\sqrt{2}}{4} \approx -0.354 \text{ and } \frac{dy}{dx} = \frac{-2}{-4\sqrt{2}} = \frac{\sqrt{2}}{4} \approx 0.354.$$

This result is consistent with what was found in Step 2.

Step 4: It can be shown that the derivatives of $f_1(x)$ and $f_2(x)$ that were found earlier are equal to the result found using the **impDif** command by rewriting the formula $\frac{dy}{dx} = \frac{-x}{y}$ strictly in terms of x :

$$\begin{aligned} \frac{dy}{dx} = \frac{-x}{y} &= \frac{-x}{\sqrt{36-x^2}} \leftarrow \frac{d}{dx}(f_1(x)) \text{ and} \\ \frac{dy}{dx} = \frac{-x}{y} &= \frac{-x}{-\sqrt{36-x^2}} = \frac{x}{\sqrt{36-x^2}} \leftarrow \frac{d}{dx}(f_2(x)). \end{aligned}$$

Problem 2 –Finding the Derivative of

Step 1: Take the derivative of both sides of $x^2 + y^2 = 36$:

$$\begin{aligned} \frac{d}{dx}(x^2 + y^2) &= \frac{d}{dx}(36) \\ \frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) &= \frac{d}{dx}(36) \\ 2x + \frac{d}{dx}(y^2) &= 0 \end{aligned}$$

The expression $\frac{d}{dx}(y^2)$ is evaluated using the **Derivative** command. Note that we use $y(x)$ instead of y .

This is a very important step when using implicit differentiation. Explain to the students that $\frac{d}{dx}(y^2) = 2y\frac{dy}{dx} = 2yy'$ and this result can be justified using the chain rule:

$$\begin{aligned} y^2 &= (f(x))^2 \rightarrow \frac{d}{dx}y^2 = \frac{d}{dx}(f(x))^2 \\ &= 2f(x)f'(x) \\ &= 2yy' \end{aligned}$$

Step 2: Students can now finish finding the derivative using implicit differentiation:

$$\begin{aligned}\frac{d}{dx}(x^2 + y^2) &= \frac{d}{dx}(36) \\ \frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) &= \frac{d}{dx}(36) \\ 2x + 2y \frac{dy}{dx} &= 0\end{aligned}$$

Solving for $\frac{dy}{dx}$:

$$\begin{aligned}2x + 2y \frac{dy}{dx} &= 0 \\ 2y \frac{dy}{dx} &= -2x \\ \frac{dy}{dx} &= \frac{-x}{y}\end{aligned}$$

This is the same result as was obtained using the **impDif** command.

Problem 3 –Finding the Derivative of

Step 1: The relation $y^2 + xy = 2$ can be solved for y using the method of completing the square. Alternatively, the **solve** command can be used to find two functions, $f_1(x)$ and $f_2(x)$, that *explicitly* define $y^2 + xy = 2$. These are

$$f_1(x) = \frac{\sqrt{x^2 + 8} - x}{2} \text{ and } f_2(x) = \frac{-\left(\sqrt{x^2 + 8} + x\right)}{2}$$

While the derivatives of $f_1(x)$ and $f_2(x)$ can be found using methods learned earlier, it should be stressed that implicit differentiation provides a more convenient method.

Step 2: Using implicit differentiation, the derivative of $y^2 + xy = 2$ can be found by hand:

$$\begin{aligned}\frac{d}{dx}y^2 + \frac{d}{dx}xy &= \frac{d}{dx}2 \\ 2yy' + xy' + y &= 0 \\ 2yy' + xy' &= -y \\ (2y + x)y' &= -y \\ \frac{dy}{dx} &= \frac{-y}{2y + x}\end{aligned}$$

Check with the students to make sure that they use the product rule to find the derivative of xy :

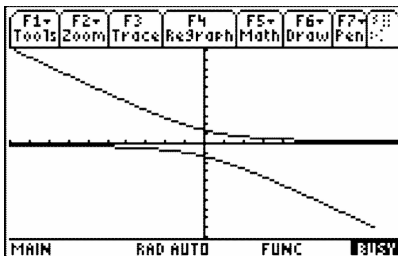
$$\begin{aligned}\frac{d}{dx}(xy) &= x \cdot \frac{d}{dx}y + y \cdot \frac{d}{dx}x \\ &= xy' + y\end{aligned}$$

Students will confirm their answer by using the **impDif** command as shown to the right.

Step 3: To check this result graphically, students should first graph the two functions:

$$f_1(x) = \frac{\sqrt{x^2 + 8} - x}{2} \text{ and } f_2(x) = \frac{-\left(\sqrt{x^2 + 8} + x\right)}{2}$$

If they let $x = -6$, then they can solve to find the y -value. Students will find two values $3 + \sqrt{11} \approx 6.32$ and $3 - \sqrt{11} \approx -0.32$. Thus, there are two points $(-6, 6.32)$ and $(-6, -0.32)$.



When they use those values in the implicit derivative $\frac{dy}{dx} = \frac{-y}{2y+x}$, they will get the slope of -0.95 for $(-6, 6.32)$ and the slope of -0.05 for $(-6, -0.32)$. Students can pick another x -value (e.g., $x = 1$) and do a similar analysis.

Extension –Finding the Derivative of

Step 1: The relation $x^3 + y^3 = 6xy$ cannot be solved explicitly for y . In this case implicit differentiation must be used.

$$\begin{aligned} \frac{d}{dx}x^3 + \frac{d}{dx}y^3 &= \frac{d}{dx}6xy \\ 3x^2 + 3y^2y' &= 6xy' + 6y \\ 3y^2y' - 6xy' &= 6y - 3x^2 \\ (3y^2 - 6x)y' &= 6y - 3x^2 \\ \frac{dy}{dx} &= \frac{2y - x^2}{y^2 - 2x} \end{aligned}$$

The **impDif** command is used to confirm this result. Students may need to do the work to see the two solutions are the same.

$$\frac{dy}{dx} = \frac{2y - x^2}{y^2 - 2x} = \frac{-x^2 + 2y}{-2x + y^2} = \frac{-(x^2 - 2y)}{-(2x - y^2)} = \frac{x^2 - 2y}{2x - y^2}$$

Step 2: Students will use $\frac{dy}{dx} = \frac{2y - x^2}{y^2 - 2x}$ to find the slope of the tangents to $x^3 + y^3 = 6xy$ at $x = 1$.

Using the **solve** command, students will find that the ordered pairs with x -coordinate equal to 1 are $(1, -2.529)$, $(1, 0.167)$ and $(1, 2.361)$.

The slopes of the tangents can now be calculated on the *Calculator* screen as shown. Students will get the following results:

$$(1, -2.52892)$$

$$m = -1.28782$$

$$(1, 0.16449)$$

$$m = 0.33728$$

$$(1, 2.36147)$$

$$m = 1.04093$$

CHAPTER

4

TE Applications of Derivatives - TI

Chapter Outline

- 4.1 XTREME CALCULUS
 - 4.2 HELICOPTER BUNGEE JUMP
 - 4.3 OPTIMIZATION
 - 4.4 LINEAR APPROXIMATION
-

4.1 Xtreme Calculus

This activity is intended to supplement Calculus, Chapter 3, Lesson 2.

ID: 11407

Time Required: 15 minutes

Activity Overview

In this activity, students will explore relative maximums and minimums by drawing tangent lines to a curve and making observations about the slope of the tangent line. This activity uses both the script feature and a program that enables the drawing of tangent lines to be animated.

Topic: Relative Extrema

- Relative Minimum
- Relative Maximum
- Critical Numbers

Teacher Preparation and Notes

- Students will need two files, *xtreme1.89t* and *tanimat2.89p*. The *xtreme1* script asks the program *tanimat2* to run. The program *tanimat2* has numerous functions that are not explored in this brief activity.
- Before beginning the activity, review with students the definitions of relative maximum, relative minimum, and critical numbers.
- To download the calculator files, go to <http://www.education.ti.com/calculators/downloads/US/Activities/Detail?id=11407> and select *main.xtreme1* and *main.tanimat2*.

Associated Materials

- Student Worksheet: Xtreme Calculus <http://www.ck12.org/flexr/chapter/9728>
- *xtreme1.89t*
- *tanimat2.89p*

Introduction

After transferring the two files *xtreme1* and *tanimat2*, students will run the script by pressing APPS, selecting the *Text Editor* application, and then opening *xtreme1*.

Students are instructed to press *F4* to advance through the script. Reading the text will provide definitions and questions similar to the student handout.

Graph 1 –Polynomial

The script sets up the window and defines a polynomial for $y_1(x)$. The *tanimat2* program uses whatever function is in $y_1(x)$ for its initial exploration. It is possible to change the function without exiting the program, but the *xtreme1* script will set up the viewing window and next function for students.



Students are instructed to explore 10 slopes. Their goal should be to find the x -value for when the slope is zero.

Students are asked to find the critical numbers of $y_1(x)$. Students are asked what occurs at each critical number of $y_1(x)$.

Solutions

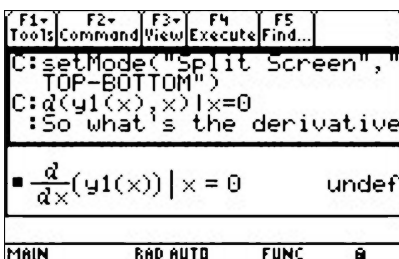
1. $x = -3, -1, 1$
2. a relative extreme value

Graph 2 –Cusp

Students are asked to find the critical numbers of $y_1(x)$. When the *tanimat2* is used to explore the slope of the tangent at the cusp, a domain error message occurs.

The script leads students in using CAS to confirm that the derivative at a cusp is undefined.

Students are asked what occurs at each critical number of $y_1(x)$.



Solutions

3. $x = 0, 2$
4. a relative extreme value

Graph 3 –Cubic

Students are asked to find the critical numbers of $y_1(x)$ and what occurs at each critical number of $y_1(x)$.

Solutions

5. $x = 1$

6. a plateau

7. A relative extreme value doesn't occur at every critical number. A critical point is only an extrema if the function changes from increasing to decreasing or decreasing to increasing, i.e. when the derivative changes sign.

Graph 4 –Negative Quadratic

Students are asked if relative extrema occur at every critical point.

Students are asked if the slope of the tangent line to the left of a relative maximum is positive, negative, or zero.

Solutions

8. positive

9. negative

Graph 5 –Positive Quadratic

Students are asked if the slope of the tangent line to the left of a relative minimum is positive, negative, or zero.

Students are asked if the slope of the tangent line to the right of a relative minimum is positive, negative, or zero.

This is the end of the script. Press ∇.

Solutions

10. negative

11. positive

Summing It All Up

Students are asked to summarize their findings about critical numbers and local extrema.

Student Solutions

12. relative maximum

13. relative minimum

14. plateau

Extension

Students are asked how many extrema an n th degree polynomial can have. They are also asked to explain their answer.

Solution

15. $n - 1$. An n th degree polynomial can only change from increasing to decreasing or vice versa $n - 1$ number of times.

4.2 Helicopter Bungee Jump

This activity is intended to supplement Calculus, Chapter 3, Lesson 4.

ID: 11760

Time Required: 40 minutes

Activity Overview

In this activity, students will observe a simulation of a record breaking bungee jump, consider a mathematical model of the height as a function of time, and take the derivative to determine points of interest like the minimum height, maximum velocity, acceleration, and maximum jerk. Students will algebraically, numerically, graphically and verbally investigate higher order derivatives.

Topic: Higher order derivatives

- Interpret the derivative in context of velocity, speed, and acceleration
- Corresponding characteristics of the graph of f, f', f''

Teacher Preparation and Notes

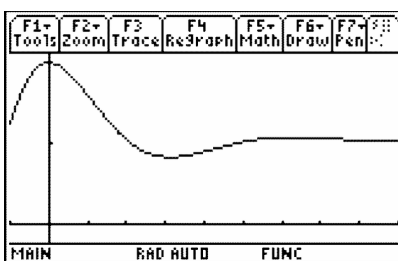
- Students will need to know that the derivative of position is velocity, and the derivative of velocity is acceleration. For the questions at the end, students are expected to be familiar with the concept of the definite integral. The Product Rule using exponential and trigonometric functions is also utilized.
- The data for the helicopter bungee jump problem is from Bungee Consultants International (<http://www.bungeeconsultants.stuntcredits.htm>).

Associated Materials

- Student Worksheet: Helicopter Bungee Jump <http://www.ck12.org/flexr/chapter/9728> , scroll down to the second activity.

Part 1 –Bungee Jump

This activity begins by describing the record breaking bungee jump. Students enter a parametric graph of the situation in their graphing calculators. If students have trouble entering the parametric equations, then direct them to change the calculator mode to parametric (press **MODE** and change **Graph** to **2:PARAMETRIC**).



On the student worksheet, students use the Product Rule to take the first and second derivatives. The derivatives can be verified on the **HOME** screen, but it is such a long function that it wouldn't fit on the screen well. To begin, some students may find it helpful for organizational purposes to use CAS to solve for $y'(t) = 0, y''(t) = 0, y'''(t) = 0, y^{IV}(t) = 0$.

The questions are designed to develop student's proficiency with CAS functionality and deepen understanding of the connection between y, y', y'', y''' .

When students are asked to enter the velocity, acceleration, and jerk functions into y_2, y_3 , and y_4 , they can use the **HOME** screen to find the derivatives of the position function and to store the functions, as shown to the right. Students can arrow up to an answer and press **ENTER** to copy an answer into the entry line.

For Question 6, many compare and contrast discoveries can be made. Question 7 points out the limitations of a mathematical model. When the jumper is in freefall for first 4 seconds or so, the only acceleration should be gravity = 32 ft/s^2 .

Solutions

1. It is expected that students will say " $v = 0$ " or " $y = \frac{\text{max}}{\text{min}}$ ". Solving when the first derivative equals zero for a position-time graph gives the time when the velocity is zero. This will give the maximum or minimum position.
2. $y''(t) = \text{acceleration}$
3. $y''(t) = 0$ when $t = 5.5 \text{ s}$, 21.2 s , and 36.9 s
4. $y^{IV}(t) = 0$ for $t > 15.7 \text{ s}$ when $t = 16.6 \text{ s}$
5. $y''(t) = 0$ when $t = 5.5 \text{ s}$. The speed is 619.98 ft/s . (Speed is scalar and should not be negative.)
6. When the height y is a minimum, the velocity graph is zero. The velocity in ft/s graph is identical to the velocity in mph except for the scale. The y, v, a , and j graphs all have a similar damped shape.
7. Maximum acceleration is about $2.5 g$ ($\frac{80 \text{ ft/s}^2}{32 \text{ ft/s}^2} = 2.5$)
8. Point of infection of $y(x)$ is $(21.218, 4684.477)$.

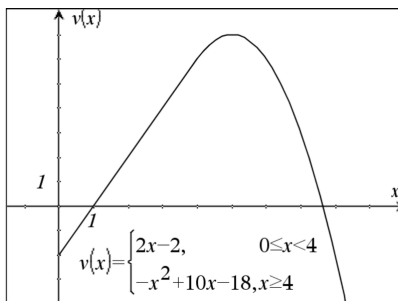
Part 2 –Graphically Examine Another Situation

To test for understanding of calculus concepts, questions relating a graph of the derivative to the original function are effective. Students do not need to know the Power Rule for integration to do the first question. They do need to understand the concept of the integral and its notation. If students know how to find the area of a triangle, they can find this integral.

The only question in this series that uses the function is Question 12—using calculus to find when v is a maximum.

For further thought and discussion

- Verify that $v(x)$ is continuous.
- Prove that $v(x)$ is differentiable.



Students will relate concavity with the graphs in Question 13. Remind students that they are finding when the graph of s is concave up, not v as shown in the graph on the student worksheet. Explaining answers “using calculus” is important to stress. Often, students justify that a function is increasing by saying “it is going up.” They should avoid the use of “it.” For example, “the derivative is positive, so the function is increasing.”

Question 14 asks when is the function s decreasing and to explain. This occurs when $s' < 0$. Since $s' = v$ and v is negative between 0 and 1, that is the solution.

For further thought and discussion

- When is s increasing?
- When is s concave up?
- When is s' concave down?

Solutions

1. $s(1) = \frac{1}{2}(1)(-2) = -1$
2. $s'(1) = v(1) = 0$
3. $s''(1) = v'(1) = \text{slope} = 2$
4. Max v occurs when $v' = 0$. Max v occurs when $v' = -2x + 10 = 0$. So $x = 5$.
5. $s'' > 0$ when $v' > 0$, i.e., when the slope of the graph is positive or when $0 < x < 5$.
6. $s' = v < 0$, when $0 < x < 1$

4.3 Optimization

This activity is intended to supplement Calculus, Chapter 3, Lesson 7.

ID: 9609

Time required: 45 minutes

Activity Overview

Students will learn how to use the second derivative test to find maxima and minima in word problems and solve optimization problems in parametric functions.

Topic: Application of Derivatives

- Find the maximum or minimum value of a function in an optimization problem by finding its critical points and applying the second derivative test. Use **Solve** (in the Algebra menu) to check the solution to $f'(x) = 0$.
- Use the command **fMin** or **fMax** to verify a manually computed extremum.
- Solve optimization problems involving parametric functions.

Teacher Preparation and Notes

- This investigation uses **FMax** and **fMin** to answer questions. Students will have to take derivatives and solve on their own.
- Before starting this activity, students should go to the home screen and select **F6 :Clean Up >2:NewProb**, then press **ENTER**. This will clear any stored variables, turn off any functions and plots, and clear the drawing and home screens.
- This activity is designed to be student-centered with the teacher acting as a facilitator while students work cooperatively.

Associated Materials

- Student Worksheet: Optimization <http://www.ck12.org/flexr/chapter/9728> , scroll down to third activity.

Problem 1 –Optimization of distance and area

Students will graph the equation $y = 4x + 7$. They need to minimize the function $s = \sqrt{x^2 + y^2}$ where x and y are the coordinates of a point on the line. The constraint is the equation of the line.

They are to rewrite the function using one variable: $s = \sqrt{x^2 + (4x + 7)^2} = \sqrt{17x^2 + 56x + 49}$.

To find the exact coordinates of the point, students will take the first derivative (**Menu >Calculus >Derivative**), and solve for the critical value (**Menu >Algebra >Solve**), and take the second derivative. Since the second derivative is always positive, there a minimum at the critical value of $x = -\frac{28}{17}$.

To find the y -coordinate, students should substitute the value of x into the original equation $y = 4x + 7$. To find the distance, they should substitute the x - and y -values into the function $s = \sqrt{x^2 + y^2}$. The point is $(-1.647, 0.412)$ and the distance is 1.698 units.

Students are to maximize the function $A = l \cdot w$. The constraint is $2l + 2w = 200$. Since $l = 100 - w$ students can rewrite the function as $A = (100 - w)w = 100w - w^2$.

Students will take the first derivative and solve to find the critical value is $w = 50$. The second derivative is always negative so we have a maximum. When $w = 50$ m, then $l = 50$ m.

The maximum area is 2500 m^2 .

Problem 2 –Optimization of time derivative problems

Remind students to use t , for time, instead of x . The position equations are the constraints.

The boat heading north is going from the right angle to the point northward. Its position equation is $y = 20t$.

The boat heading west is going to the right angle. At 1 pm, it is one hour from the arrival time 2 pm so it is 15 km away. Its position equation is $x = 15 - 15t$.

Students are to minimize the distance function $s = \sqrt{x^2 + y^2} = \sqrt{(15 - 15t)^2 + (20t)^2}$.

There is a restriction of $0 < t < 1$ because the boats are only moving for 1 hour. Students will solve the first derivative to find the critical time is $t = \frac{9}{25}$. Since the second derivative is always positive, there is a minimum.

The time at which the distance between the boats is minimized is $\left(\frac{9}{25}\right) \cdot 60 = 21.6$ minutes after 1 pm or about 1:22 pm. The distance between the two boats is 12 km.

Extension –Parametric Function

To rewrite the parametric equations, students will need to know that $\sin(30^\circ) = 0.5$ and $\cos(30^\circ) = \frac{\sqrt{3}}{2}$.

To find when the projectile hits the ground, students are to set $y = 0$ and solve. ($t = 0$ and $t = 51.02$). Substituting these values into the x function gives how far away it lands (22,092.5 units). Students can find the maximum height when $\frac{dy}{dt} = 0$ ($t \approx 25.51$). Substituting this value in the function for y students should get 3188.78 units high.

4.4 Linear Approximation

This activity is intended to supplement Calculus, Chapter 3, Lesson 8.

ID: 9470

Time required: 45 minutes

Activity Overview

In this activity, students will graph the functions, construct the tangent line at a point, and find an estimate from a linear approximation. They will then determine whether the estimate is an overestimate or an underestimate, and find an interval for a desired accuracy.

Topic: Application of Derivatives

- Calculate the equation of the tangent line to a graph at any given point.
- Construct a tangent line to the graph of a differentiable function at $x = a$ to approximate its value near $x = a$.
- Construct a tangent line to the graph of a differentiable function at $x = 0$ to approximate its value near $x = 0$.

Teacher Preparation and Notes

- This investigation offers an opportunity for students to develop an understanding of how a tangent line to a curve can be used to approximate the values of a function near the point of tangency. Linear approximations are often used in scientific applications, including formulas used in physics where $\sin\theta$ is replaced with its linear approximation, θ .
- This activity is designed to be **student-centered** with the teacher acting as a facilitator while students work cooperatively. The student worksheet is intended to guide students through the main ideas of the activity and provide a place for them to record their observations.
- The students will need to be able to enter the functions and use the commands on their own.
- Before starting this activity, students should go to the home screen and select **F6:Clean Up >2:NewProb**, then press **ENTER**. This will clear any stored variables, turn off any functions and plots, and clear the drawing and home screens.

Associated Materials

- Student Worksheet: Linear Approximation <http://www.ck12.org/flexr/chapter/9728> , scroll down to the fourth activity.

Part 1 –Introduction

From top to bottom, the points where the horizontal lines cross the y -axis are $L(x)$, $f(x)$, $f(a)$.

$L(x)$ will be the linear approximation.

$L(x) - f(x)$ = error of this estimate.

Since $L(x) > f(x)$, this estimate is an overestimate.

To get the horizontal line, students can use **graph > F7 :Pencil >5:Horizontal**

To get the vertical line, they can use **graph > F7 :Pencil >6:Vertical**.

Part 2 –Investigating linear approximation

Students are to graph the function of $f_1(x) = x^3 - 3x^2 - 2x + 6$ and the tangent at $a = -1$. The point p is where the vertical line crosses the function. The point q is where the vertical line crosses the tangent line.

For the x -value 0.2:

$L(0.2) = 12.4$ is the linear approximation.

The error is the length $pq = 6.912$ ($12.4 - 5.488$).

The true value, $f(0.2) = 5.488$. It is an overestimation.

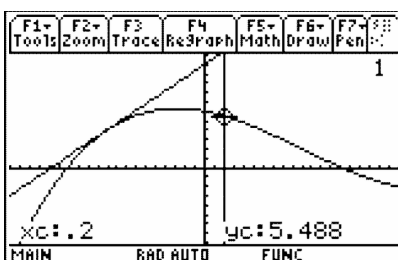


TABLE 4.1:

	linear approx. of $f_1(q)$	real value of $f_1(q)$	error	underestimation/overestimation
$x = -0.2$	9.6	6.272	3.328	overestimation
$x = -0.5$	7.5	6.125	1.375	overestimation
$x = -0.6$	6.8	5.904	0.896	overestimation
$x = -1.2$	2.6	2.352	0.248	overestimation

As you get close to the point of tangency, the graph of the function and the graph of the tangent line appear to be the same.

They are called local linearization because the graph acts like a straight line at the point of tangency and that line is the tangent line.

Students are to find the derivative of $f_1(x) = x^3 - 3x^2 - 2x + 6$ and evaluate it at $x = -1$. They should use the slope and the point $(-1, 4)$ to get the equation of the line.

$$y - 4 = 7(x + 1) \rightarrow y = 7x + 11$$

$$L(x) = 7x + 11$$

$L(-1.03) = 3.79$. This is the linear approximation.

$$\text{Error} = 3.79 - 3.78457 = 0.005427$$

Part 3 –Underestimates versus overestimates

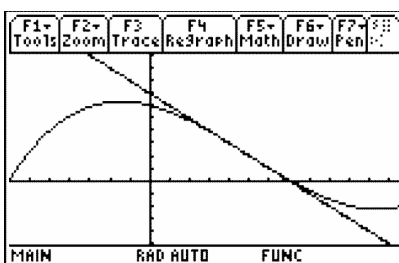
Students are to graph $f_1(x) = x^3 - 3x^2 - 2x + 6$ and place a tangent line $a = 1$.

If p is to the left of $a = 1$, then the tangent line will be above the function and the linear approximation is an overestimate.

If p is to the right of $a = 1$, then the tangent line will be below the function and the linear approximation is an underestimate.

The point $a = 1$ is a point of inflection. Students can look at the second derivative and set it equal to zero to confirm this.

Students should see that in general the linear approximation will overestimate if the curve is concave down and it will underestimate when the curve is concave up.



Part 4 –Finding intervals of accuracy

Students are posed with the question: *How close to -1 must x be for the linear approximation to be within 0.2 units of the true value of $f_1(x)$?*

When students look at the graph, they want the tangent line to be within the bound of $f_1(x) + 0.2$ and $f_1(x) - 0.2$.

Since the linear approximation overestimates in this region, they want to compare $L(x)$ and $f_1(x) + 0.2$.

Students can zoom in graphically or solve algebraically. The interval is $(-1.1799, -0.81453)$ using the **solve** command.

Because the tangent line overestimates to the left of $x = 1$ and underestimates to the right of $x = 1$ students will have to do this in two parts.

Since the linear approximation overestimates in the region to the left of $x = 1$, students need to compare $L(x)$ and $f_1(x) + 0.2$ there.

They can zoom in graphically or solve algebraically. The interval is $(0.415196, 1)$ using the **solve** command.

Since the linear approximation underestimates in the region to the right of $x = 1$, students want to compare $L(x)$ and $f_1(x) - 0.2$ there.

They can zoom in graphically or solve algebraically. The interval is $(1, 1.5848)$ using the **solve** command.

CHAPTER

5

TE Integration - TI

Chapter Outline

5.1 **EXPLORING THE AREA UNDER A CURVE**

5.2 **SUM RECTANGLES**

5.3 **FTC CHANGED HISTORY**

5.1 Exploring the Area Under a Curve

This activity is intended to supplement Calculus, Chapter 4, Lesson 3.

ID: 9735

Time required: 45 minutes

Activity Overview

In this activity, students will explore Riemann sums to find the approximate area under the graph of $y = x^2$ on the interval $[0, 1]$. Additionally, students will learn to write formulas in expanded form and using summation notation. Finally, they will be asked to summarize their findings, including an analysis of when approximations produce overestimates or underestimates and the idea that an infinite number of approximating rectangles will yield the exact area under the curve.

Topic: The Fundamental Theorem of Calculus

- *Approximate the area under a curve by constructing a Riemann sum and calculating its sum.*

Teacher Preparation and Notes

- *It is essential to review the concept of summation notation with students beforehand.*
- *Students should know how to write an arithmetic sequence which will be part of the summation expression they will use to find approximate areas.*
- *The students will need to be able to compute the areas and heights on their own.*

Associated Materials

- Student Worksheet: Exploring the Area under a Curve <http://www.ck12.org/flexr/chapter/9729>

Problem 1 –Explore and discover

Two focus questions define this activity:

- How can you use rectangles to approximate the area under the curve $y = x^2$ and above the x -axis?
- Is there a way to use rectangles to find the exact area under the curve?

Some discussion should follow the posing of these questions, mainly related to how these rectangles should be used. Prior experience with limits will suggest that a large number of rectangles will produce a more accurate estimate. Tell the students that, for this activity, they will look at exactly five rectangles.

The student worksheet has two graphs for the students to try drawing rectangles.

Problem 2 –Using five right endpoint rectangles

When the students divide the interval into five equal pieces, they should get the intervals $[0, 0.2]$, $[0.2, 0.4]$, $[0.4, 0.6]$, $[0.6, 0.8]$, and $[0.8, 1.0]$. The students are to draw the rectangles on their graph. They can use the Trace feature to determine the heights of the rectangles.

The area for each rectangle is $0.02 \times \text{height of the rectangle}$.

TABLE 5.1:

Interval	Right Endpoint	Height	Area
$[0, 0.2]$	0.2	0.04	0.008
$[0.2, 0.4]$	0.4	0.16	0.032
$[0.4, 0.6]$	0.6	0.36	0.072
$[0.6, 0.8]$	0.8	0.64	0.128
$[0.8, 1.0]$	1.0	1.0	0.2

To find the approximation of the area under the curve, students are to use the formula given on the worksheet and then also find the sum of the last column. They should get the same answer $R_5 = 0.44$.

Then, students are to write the sum of the rectangles in Sigma notation.

Since students need to sum from 1 to 5, they have to adjust the endpoints. Each endpoint is 0.2 times the index number. To factor in the width of the interval, they need to have the $\times 0.2$ at the end of the summation. The sigma command can be found in the Calc menu.

Problem 3 –Using left endpoint rectangles

Students are to draw the rectangles on their graph using left-endpoints and then complete the table.

The area for each rectangle is $0.02 \times \text{height of the rectangle}$.

TABLE 5.2:

Intervals	Left endpoints	Height	Area
$[0, 0.2]$	0	0	0
$[0.2, 0.4]$	0.2	0.04	0.008
$[0.4, 0.6]$	0.4	0.16	0.032
$[0.6, 0.8]$	0.6	0.36	0.072
$[0.8, 1.0]$	0.8	0.64	0.128

To find the approximation of the area under the curve, students are to use the formula given on the worksheet and then also find the sum of the last column. They should get the same answer, $L_5 = 0.24$.

When students use the sigma notation, they have the option of summing from 0 to 4 as in the top screen or adjusting the number and keeping the limits the same. Both appear at the right.

Problem 4—Using five midpoint rectangles

Students are to draw the rectangles on their graph using the midpoints and then complete the table

The area for each rectangle is $0.02 \cdot \text{height of the rectangle}$.

TABLE 5.3:

Interval	Midpoint	Height	Area
[0, 0.2]	0.1	0.01	0.002
[0.2, 0.4]	0.3	0.09	0.018
[0.4, 0.6]	0.5	0.25	0.05
[0.6, 0.8]	0.7	0.49	0.098
[0.8, 1.0]	0.9	0.81	0.162

To find the approximation of the area under the curve, students are to use the formula given on the worksheet and then also find the sum of the last column. They should get the same answer, $M_5 = 0.33$.

The sigma notation is $\sum_{x=1}^5 (0.2 \cdot x - 1)^2 \cdot 0.2$.

Discuss with students about whether they think that this is an overestimation or underestimation of the true area.

Emphasize the similarities and difference between these calculations and the previous ones in Problems 2 and 3.

Problem 5 –Summarize your findings

Students should readily agree that the midpoint approximation produces the best answer.

Furthermore, students should respond that the concavity of the function and whether it is increasing or decreasing will determine which estimates are *overestimates* versus *underestimates*. In this activity, students were working with a region that was determined by a function that was concave up and increasing. In this type of situation, right-endpoint and midpoint approximations will always give *overestimates* and left-endpoint approximations will always yield *underestimates*.

Finally, students will observe that a large number of approximating rectangles will improve upon their answers. Additionally they will see that summation notation offers a convenient method of quickly evaluating a large number of rectangles. The screens to the right show the summation for each method with 500 approximating rectangles.

As an extension to this activity, you may decide to use the limit and evaluate these sums for an infinite number of rectangles.

F1 Tools	F2 Algebra	F3 Calc	F4 Other	F5 PrfMod	F6 Clean Up
$\sum_{x=1}^{500} \left(\left(\frac{1}{500} \cdot x \right)^2 \right) \cdot 1$					
500				167167	500000
$\dots / 500 \cdot x \wedge 2, x, 1, 500) \cdot 1 / 500$					
MAIN	RAD AUTO	FUNC	1/30		

5.2 Sum Rectangles

This activity is intended to supplement Calculus, Chapter 4, Lesson 4.

ID: 12099

Time Required: 15 minutes

Activity Overview

In this activity, students will graphically and numerically explore Riemann sums and develop an understanding of summation notation for adding these rectangles.

Topic: Riemann sums

- *Graphically exploring Riemann sums*
- *Summation notation*

Teacher Preparation and Notes

- *The file area.89p is a program file. It contains several approximation methods using Riemann sums, the Trapezoidal Rule, and Simpson's Rule. This program should be sent to student graphing calculators prior to beginning this activity*
- *To download the calculator file, go to <http://www.education.ti.com/calculators/downloads/US/Activities/Detail?id=12099> and select main.area.89p*

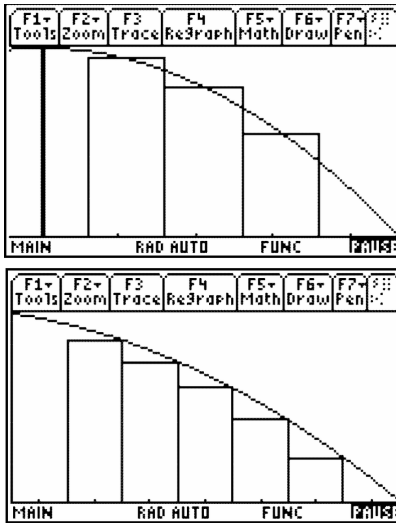
Associated Materials

- Student Worksheet: Sum Rectangles <http://www.ck12.org/flexr/chapter/9729> , scroll down to the second activity.
- main.area.89p

Part 1 –Graphical Riemann Sums

Students will use the program **area** to complete this part of the activity. To run the program enter **area()** on the Home screen. To use the program, select to view the approximation either graphically or numerically. Next, enter the equation that you are examining and press \div . Enter the minimum value, maximum value, and the number of steps. Then, select the approximation method and press \div . The approximation will be displayed on the screen. Then, select to enter another function, do another approximation or to exit the program.

Many questions can be asked and observations made. This section can be extended by asking about other functions. Students should graph other functions that have different concavity or slope to complete Exercise 3.



Students get a visual of what Riemann sums are. If only a few rectangles are graphed students may be able to better see and understand why the midpoint is an over approximation in some situations and an under approximation at other times.

Student Solutions

1. The midpoint approximation is between the left and right endpoint approximations. The left is too big and the right too small.
2. The Riemann sums converge (to the definite integral) as $n \rightarrow \infty$. With thinner widths, the rectangles approach the true area.
3. (a) over, (b) under, (c) over, (d) under. For example, when the function is decreasing and concave down, the function curves more steeply for the second half of the rectangle than the first. Therefore, since there is more rectangle above than below the graph, it over estimates the area.

Part 2 –Summation notation

In Part 2, students examine the summation notation. By reading and answering questions on the student worksheet, students will better understand the sigma notation.

To make entering the summation easier, encourage students to define the function before entering the sum. To enter a summation on the **HOME** screen, students should select **F3 :Calc >4:Σ (sum**. The syntax for the summation function is as follows:

$\Sigma(\text{expression, variable, low, high})$.

Student Solutions

1. $\Delta x = h = 1$
2. $y(a) + y(a+1) + y(a+2) + y(a+3) + y(a+4)$
3. The right Riemann sums from a to b with 5 subintervals, looks like $y(a+1) + y(a+2) + y(a+3) + y(a+4) + y(a+5)$. Note $a+5 = b$ since $\Delta x = 1$
4. $\sum_{i=1}^n (h \cdot y(a + (i-1)h)) | h = \frac{b-a}{n}$

When $n = 10$, Left Riemann sum = 168.4375

When $n = 20$, Left Riemann sum = 166.328125

When $n = 50$, Left Riemann sum = 165.0375

When $n = 100$, Left Riemann sum = 164.603125

Extension –Area Programs

Students will use the **area** program to compare area approximation methods.

Student Solutions

1.

$n = 10$	$n = 50$	$n = 100$
mid = 71.5625	mid = 71.6625	mid = 71.6656
trap = 71.875	trap = 71.675	trap = 71.6688

2. For $y = x^2$, an increasing concave up function in this domain, the midpoint is approaching $71\frac{2}{3}$ from the left and the trapezoid method is approaching the value of the definite integral from the right. Midpoint gives a better approximation.

5.3 FTC Changed History

ID: 9778

Time required: 45 minutes

Activity Overview

This activity builds student comprehension of functions defined by a definite integral, where the independent variable is an upper limit of integration. Students are led to the brink of a discovery of a discovery of the Fundamental Theorem of Calculus, that $\frac{d}{dx} \int_0^x f(t) dt = f(x)$.

Topic: Fundamental Theorem of Calculus

- Graph a function and use *Measurement >Integral* to estimate the area under the curve in a given interval.
- Use *Integral* (in the *Calculus* menu) to obtain the exact value of a definite integral.

Teacher Preparation and Notes

- This investigation should follow coverage of the definition of a definite integral, and the relationship between the integral of a function and the area of a region bounded by the graph of a function and the x -axis.
- Before doing this activity, students should understand that if $a < b$ and $f(x) > 0$, then:

$$\begin{aligned} & - \int_a^b f(x) > 0 \\ & - \int_b^a f(x) < 0 \\ & - \int_a^a f(x) < 0 \\ & - \int_b^b f(x) < 0 \\ & - \int_a^b -f(x) > 0 \end{aligned}$$

- Before starting this activity, students should go to the **HOME** screen and select **F6 :Clean Up >2:NewProb**, then press **ENTER**. This will clear any stored variables.

Associated Materials

- Student Worksheet: FTC Changed History <http://www.ck12.org/flexr/chapter/9729> , scroll down to the third activity.

Problem 1 –Constant integrand

Students explore the function $\int_0^x 1.5 dt$. They should notice that there is a constant rate of change in the graph of

$$f(x) = \int_0^x 1.5 dt \text{ This rate of change is } 1.5.$$

1. The table looks like the one below.

TABLE 5.4:

x	$\int_0^x 1.5 \, dt$
1	1.5
2	3
3	4.5
4	6
5	7.5

2. $\int_0^0 1.5 \, dt = 0$; There is zero area under the graph of $y = 1.5$ from $x = 0$ to $x = 0$.

3. 1.5 units

4. The graph will be a line through the origin with slope 1.5.

Students will enter their data into the lists and then view the graph of $\left(x, \int_0^x 0.5 \, dt\right)$.

5. A line; yes (student answers may vary)

6. The same as before, except the slope would be 0.5 instead of 1.5.

Problem 2 –Non-Constant Integrand

Students investigate the behavior of $f(x) = \int_0^x \frac{t}{2} \, dt$. Students should note that this function changes at a non-constant rate and are asked to explain why this is so (from a geometric point of view).

TABLE 5.5:

x	$\int_0^x \frac{t}{2} \, dt$
1	0.25
2	1
3	2.25
4	4
5	6.25

8. $\int_0^0 \frac{t}{2} \, dt = 0$ The height and the length of the triangle are 0 so the area is 0.

9. The area changes by a different amount each time because both the height and width are increasing.

10. The graph is not linear. It is a parabola as seen by the formula in the above screen shot.

Problem 3 –An Integrand That Changes Sign

TABLE 5.6:

x	$\int_0^x \frac{t^2 - 13t + 22}{9} dt$
1	$\frac{95}{54} = 1.76$
2	$\frac{62}{27} = 2.29$
3	$\frac{11}{6} = 1.83$
4	$\frac{16}{27} = 0.59$
5	$-\frac{65}{54} = -1.20$
6	$-\frac{10}{3} = -3.33$
7	$-\frac{301}{54} = -5.57$
8	$-\frac{208}{27} = -7.70$
9	$-\frac{19}{2} = -9.50$
10	$-\frac{290}{27} = -10.74$
11	$-\frac{605}{54} = -11.20$
12	$-\frac{32}{3} = -10.67$
13	$-\frac{481}{54} = -8.91$
14	$-\frac{154}{27} = -5.70$

The fractions are approximated to the nearest hundredth.

11. After $x = 2$, the integral value begins to decrease.

12. The values for x in which the integral decreases are $2 < x < 11$; the function is negative.

13. The values for which the integral is increasing are $x < 2, x > 11$; the function is positive.

14. The table seems to indicate $x = 11$. To find out for sure, use **fMin** on the integral. You have to restrict the domain to $x > 0$, to get the answer.

15. Yes, we have seen a similar situation. The minimum occurs on a function where the function stops decreasing and starts increasing.

CHAPTER

6

TE Applications of Integration

Chapter Outline

- 6.1 THE AREA BETWEEN
 - 6.2 VOLUME BY CROSS SECTIONS
 - 6.3 GATEWAY ARC LENGTH
-

6.1 The Area Between

This activity is intended to supplement Calculus, Chapter 5, Lesson 1.

ID: 9983

Time required: 45 minutes

Topic: Applications of Integration

- Calculate the area enclosed by two intersecting curves defined in Cartesian coordinates.
- Use the *Solve* and *Integral* commands to verify the manual computation of areas bounded by curves.

Activity Overview

In this activity, students will use the TI-89 graphing calculator to find the area between two curves while determining the required amount of concrete needed for a winding pathway.

Teacher Preparation

The students should already be familiar with the concept of the integral as well as using them to find the area below a curve.

- Students should know how to use the **Integral** and **solve** commands.
- The screenshots on page 1 demonstrate expected student results.

Classroom Management

- This activity is designed to be **student-centered**. You may use the following pages to present the material to the class and encourage discussion. Students will follow along using their calculators.
- The students will need to be able to compute the integrals and solve equations using their calculator on their own. You may choose to have students compute these by hand.
- Before starting this activity, students should go to the home screen and select **F6 :Clean Up >2:NewProb**, then press \div . This will clear any stored variables, turn off any functions and plots, and clear the drawing and home screens.

Associated Materials

- Student Worksheet: The Area between Two Curves <http://www.ck12.org/flexr/chapter/9730>

Problem 1 –Making Pathways

The first problem involves a pathway with sine functions as the borders. Students are to graph the functions in y_1 and y_2 . Then they can use the **Integral** tool to calculate the area under each curve. Students will be asked to enter the lower and upper limits. They can enter -2π and 2π directly, instead of making a guess with the arrow keys.

Students are to go to the Home screen, where they will calculate the area between the curves using the **nInt** command. Then they need to multiply their answer by $\frac{1}{3}$ to find the volume of the pathway or amount of concrete needed (12.5664 ft^3).

Problem 2 –Finding New Pathways

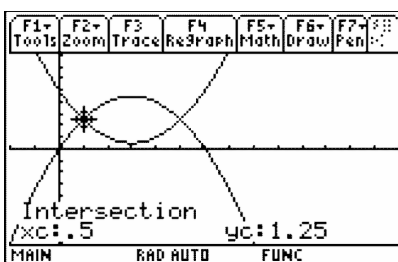
The second problem involves a pathway with cubic functions as the borders. Students are to find the volume of a new pathway using the graph and the calculator. They should use the same method in the previous problem.

Remind students that when finding the area of the region, they must always take the integral of the top function minus the bottom one (this is where many mistakes happen).

Problem 3 –Stepping Stones

In this problem, students will find the volume of one stepping stone. Students are to graph the functions and then find the intersection points using the **Intersection** tool. These intersection points will be the lower and upper limits. Students can also use the **Solve** command on the Home screen; however it will give three values. They need to use the two positive values.

Then, students can find the volume of the stepping stone using the previous method. They may think that $g(x)$ is the top function, but $f(x)$ is the top function in the interval.



6.2 Volume by Cross Sections

This activity is intended to supplement Calculus, Chapter 5, Lesson 2.

ID: 12280

Time Required: 15 minutes

Activity Overview

In this activity, students will be introduced to the concept of finding the volume of a solid formed by cross sections of a function that form certain shapes. Since volume is the area of the base times the height and $dV = \text{Area} \cdot dx$, students review areas of various shapes like squares, semicircles and equilateral triangles. Calculator screenshots are used to help students get a visual of the volume under consideration. Students will practice what they learn with exam-like questions.

Topic: Volume by Cross Sections

- Applications of integration
- Volume by cross sections

Teacher Preparation and Notes

- Part 1 of this activity takes less than 15 minutes. Part 2 contains three exam-like questions that have accompanying visuals that can be used as an extension or homework.
- Students will write their responses on the accompanying handout where space is provided for students to show work when applicable.

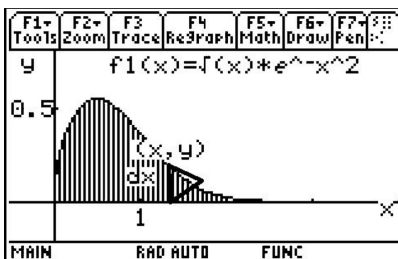
Associated Materials

- Student Worksheet: Volume by Cross-Section <http://www.ck12.org/flexr/chapter/9730> , scroll down to the second activity.

Part 1 –Setting Up The Problem And Understanding The Concept

In this section students are introduced to the concept of finding the volume of a solid formed by cross sections of a function that form certain shapes. Since volume is the area of the base times the height and $dV = \text{Area} \, dx$, students review areas of various shapes like squares, semicircles and equilateral triangles.

Part 1 ends with students finding the volume with equilateral triangle cross sections.



Student Solutions

1. dx
 - a. base times height. The area of a square with side x is x^2 .
 - b. $\frac{1}{2}\pi r^2$
2. $\frac{1}{2}y\frac{\sqrt{3}}{2}y$
3. 0.433013 cm^2
- 4.

$$\begin{aligned}\int_0^2 \frac{1}{2}y\frac{\sqrt{3}}{2}y \, dx &= \int_0^2 \frac{1}{2} \left(\sqrt{x} \cdot e^{-x^2} \right) \frac{\sqrt{3}}{2} \left(\sqrt{x} \cdot e^{-x^2} \right) dx \\ &= \int_0^2 \frac{\sqrt{3}}{4} x \cdot e^{-2x^2} dx\end{aligned}$$

If students use u -substitution, $u = -2x^2$, $du = -4x \, dx$ and the limits of integration are from 0 to -8.

$$-\frac{\sqrt{3}}{16} \int_0^{-8} e^u du = -\frac{\sqrt{3}}{16} (e^{-8} - 1) = \frac{\sqrt{3}}{16} \left(1 - \frac{1}{e^8} \right)$$

Part 2 –Homework

This section enables students to get a visual of challenging exam-like questions. Students should show their work on the first two questions and show their set up on the third question.

Student Solutions

1. $\frac{3\pi}{32} \text{ units}^3$
2. 2 units^3
3. 1.57 units^3

6.3 Gateway Arc Length

This activity is intended to supplement Calculus, Chapter 5, Lesson 3.

ID: 12439

Time Required: 15 minutes

Activity Overview

*Students will investigate the arc length of the Gateway Arch. They will use the Pythagorean Theorem to approximate and use Calculus to find the exact solution. They will also use CAS capabilities, including **arcLen**(), to solve a variety of arc length questions.*

Topic: Differential Equations

- Arc length approximation, calculus formula, and using CAS
- Find the arc length of parametric equation

Teacher Preparation and Notes

- Arc length is a Calculus BC topic. Calculus AB teachers may enjoy using this activity after the AP* exam or using with students in your AB class who want to prepare for the BC exam. After completing the activity, students should be more successful with AP questions like multiple choice 03BC15, 98BC21, 88BC#33, and free response 04formB BC1c, 02formB BC1d3c, 01BC1c, 97BC1e3b. For four of these six free response questions the graph is given in parametric form.
- The syntax for **arcLen** is **arcLen**($f(x)$, x , a , b) where $f(x)$ is the function, x is the variable and the arc length is to be found from $x = a$ to $x = b$. This activity will help students approximate arc length and use calculus to find the exact arc length.

Associated Materials

- Student Worksheet: Gateway Arch Length <http://www.ck12.org/flexr/chapter/9730> , scroll down to the third activity.

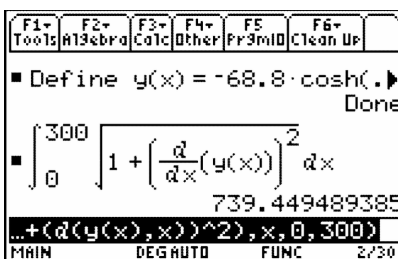
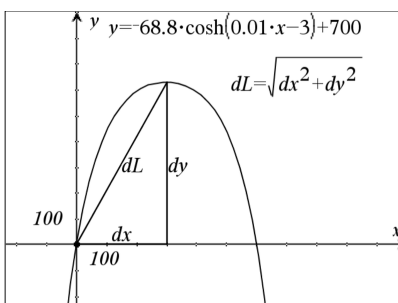
* AP, College Board, and SAT are registered trademarks of the College Board, which was not involved in the production of and does not endorse this product.

Part 1 –Arc Length Introduced

The first question investigates the Gateway Arch and the distance that one would travel if they rode the elevator tram to reach the top. The Pythagorean Theorem is used to approximate the distance. The graph on the student worksheet helps students visually understand why those numbers were used in the solution to Exercise 1. The formula for arc length is derived from the Pythagorean Theorem. To help students understand why the integral formula approximates the arc length, compare this method to finding the area under a curve using a Riemann Sum. When the infinitesimal values of dL are added together from a to b the arc length is found.

Discussion Questions

- What are the conditions for which the Pythagorean Theorem applies? *If students say, “It works for triangles,” press them further. What are characteristics of a triangle? Perhaps they will see then, “Oh yeah, the Pythagorean Theorem only works for right triangles. If you would like, you could go a bit deeper and ask, “What relationship (principle or law) applies for triangles that are not right? Explain it.” Law of Cosines $c^2 = a^2 + b^2 - 2ab \cos \theta$ where θ is the angle between a and b . The Pythagorean Theorem is a special case of this where $\theta = 90^\circ$. (They may also say Law of Sines.)*



- Ask students, “Remind me, what is an integral? What does it mean?” *You may need to remind them that the definition was based on the area of rectangles or Riemann sums. Ask again, “What does this mean? What are you doing with Riemann Sums?” Adding infinite infinitesimals.*

For Exercise 2, students are to use CAS to find the arc length of the Gateway Arch equation. Have students store the equation modeling the arch before entering the arc length formula. Students compare this solution with their length from Exercise 1.

Arc length for parametric equations is introduced and students are to solve this arc length by hand. For Exercise 4, students use CAS to find the arc length for the function $y = \sqrt{4 - x^2}$ from $x = 0$ to $x = 2$. Students should make the connection that this curve is also a fourth of a circle with radius 2.

The **arcLen** command is also introduced.

Student Solutions

1. The distance is at least $704 = \sqrt{315^2 + 630^2}$. The curve will be longer than the straight line connecting the base to the peak.

$$2. \int_0^{300} \sqrt{1 + \left(\frac{d}{dx}(f1(x))\right)^2} dx = 739.449$$

This is reasonable since it is a little larger than the straight line found in Exercise 1.

3.

$$\begin{aligned}
 & \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\
 &= \int_0^{\frac{\pi}{2}} \sqrt{\left(\frac{d}{dt} 2\cos(t)\right)^2 + \left(\frac{d}{dt} 2\sin(t)\right)^2} dt \\
 &= \int_0^{\frac{\pi}{2}} \sqrt{(-2\sin(t))^2 + (2\cos(t))^2} dt \\
 &= \int_0^{\frac{\pi}{2}} \sqrt{4(\sin^2(t) + \cos^2(t))} dt = 2t \Big|_0^{\frac{\pi}{2}} = \pi
 \end{aligned}$$

$$4. \int_0^2 \sqrt{1 + \left(\frac{d}{dx}(\sqrt{4-x^2})\right)^2} dx \approx 3.14159$$

5. The solution will be around 10.

$$\int_0^3 \sqrt{1 + \left(\frac{d}{dx}(x^2 - 9)\right)^2} dx \approx 9.747$$

6. The arc length will be more than 5, because $5 = \sqrt{3^2 + 4^2}$

$$\int_0^3 \sqrt{1 + \left(\frac{d}{dx}\left(-x^2 + \frac{5}{3}x + 4\right)\right)^2} dx \approx 6.492$$

This answer is larger than 5 which is expected.

Part 2 –Additional Practice

Students are expected to know the arc length formula and answer multiple-choice questions without a calculator. Question 2 is a parametric arc length question.

Student Solutions

$$1. D) \int_a^b \sqrt{\frac{x^2 - 5}{x^2 - 4}} dx$$

$$2. A) \int_0^{\pi} \sqrt{\cos^2 t + 1} dt$$

$$3. E) \int_a^b \sqrt{1 + \sec^4 x} dx$$

CHAPTER

7

TE Transcendental Functions - TI

Chapter Outline

7.1 INVERSES OF FUNCTIONS

7.2 THE LOGARITHMIC DERIVATIVE

7.1 Inverses of Functions

This activity is intended to supplement Calculus, Chapter 6, Lesson 1.

ID: 11405

Time Required: 45 minutes

Activity Overview

In this activity, students will explore three ways to find the inverse of a function. First, students graph two scatter plots and find the line of reflection. Then, they will graph a line and use the x - and y - intercepts to create the graph of the inverse. Students will also find the inverse of the line algebraically.

Topic: Sequences, Series Functions

- Reflection
- Inverse of data points
- Inverse of functions

Teacher Preparation and Notes

- Before beginning this activity, students should clear lists L_1 and L_2 and turn off all functions and plots.
- Students will need to know how to find the midpoint of two points. The formula is given later in this document.

Associated Materials

- Student Worksheet: Inverses of Functions <http://www.ck12.org/flexr/chapter/9731>

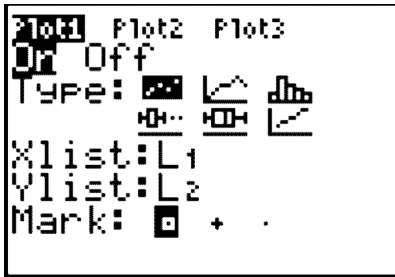
Exploring the Problem

Students are given data for a wind tunnel experiment. They are to use the data to create a scatter plot, then answer questions about the data and associated graph.

L1	L2	L3	Z
10	6.4	-----	
21	9.2		
34	17		
40	22.4		
45	30.2		
55	59.2		
-----	-----		
L2(7) =			

Students begin by constructing a scatter plot and drawing their graph on their worksheet. To create a scatter plot, students first enter the data into their lists by pressing **STAT** and selecting **Edit**.

Once the data is entered, students should press $2^{nd}Y=$ and select **Plot1**. Pressing **ZOOM** and selecting **ZoomStat** will set a window that displays all of the data.



Students will switch $L1$ and $L2$ for the Xlist and Ylist and create a new scatter plot in **Plot2**. They are asked to comment on their observations of the two scatter plots on the worksheet.

The points obtained by switching the domain and range appear to be a reflection of the original points across a line.

Students are instructed to find the midpoint between the first point on each of the scatter plots and the midpoint between the last points for each of the scatter plots on the graph.

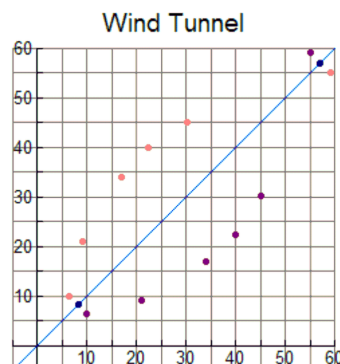
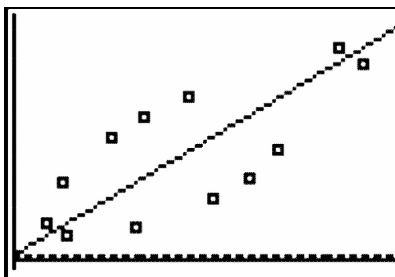
$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

Then, to find the equation of the line, they need to find the slope using the two midpoints and the **point-slope form**.

4.

$$\begin{aligned} \left(\frac{10 + 6.4}{2}, \frac{6.4 + 10}{2} \right) &= \left(\frac{16.4}{2}, \frac{16.4}{2} \right) = (8.2, 8.2) \\ \left(\frac{55 + 59.2}{2}, \frac{55 + 59.2}{2} \right) &= \left(\frac{114.2}{2}, \frac{114.2}{2} \right) = (57.1, 57.1) \end{aligned}$$

5. $y = x$; This line represents the reflection line.



Developing the Pattern Further

At this point, students should clear out lists $L1$ and $L2$ and turn off all functions and plots.

Students are to graph $y = 2x + 3$ in a standard viewing window. To find the x -intercept, they will use the **zero** function. When selected, it asks for the lower bound, upper bound, and a guess. There must be a sign change in order for this command to work. The y -intercept can be found by just pressing **TRACE**.

Then, students will switch the locations of the x - and y -coordinates of these points to (y, x) , input these as $L1$ and $L2$, and graph using **Plot1**. They can find the equation of the line connecting these two points by finding the slope and then using the **slope-intercept form**. They will graph it in $Y2$.

Students and teachers are encouraged to explore the concept of inverses further. The necessity of a function being one-to-one in order to have an inverse should be addressed by the teacher.

6. $(0, 3)$ and $(-1.5, 0)$

7. $y = \frac{(x-3)}{2}$ or $y = \frac{x}{2} - \frac{3}{2}$ or $y = 0.5x - 1.5$

8. It is the same.

9. Switch the domain and range, switch x and y in the equation, reflect the graph of the function across the line $y = x$.

10. Agree; Sample Answer: In order for a function to have an inverse, it must be one-to-one. In other words, it passes both the vertical and horizontal line tests, which will result in a reflection across $y = x$ that will also pass the vertical line test, and is therefore a function.

11. a. $f^{-1}(x) = \frac{x+2}{6}$

b. $f^{-1}(x) = 2x + \frac{3}{2}$

c. $f^{-1}(x)$ does not exist

7.2 The Logarithmic Derivative

This activity is intended to supplement Calculus, Chapter 6, Lesson 3.

ID: 9092

Time required: 45 minutes

Activity Overview

Students will determine the derivative of the function $y = \ln(x)$ and work with the derivative of both $y = \ln(u)$ and $y = \log_a(u)$. In the process, the students will show that $\lim_{h \rightarrow 0} \frac{\ln(a+h) - \ln(a)}{h} = \frac{1}{a}$.

Topic: Formal Differentiation

- *Derive the Logarithmic Rule and the Generalized Logarithmic Rule for differentiating logarithmic functions.*
- *Prove that $\ln(x) = \ln(a) \cdot \log_a(x)$ by graphing $f(x) = \log_a(x)$ and $g(x) = \ln(x)$ for some a and deduce the Generalized Rule for Logarithmic differentiation.*
- *Apply the rules for differentiating exponential and logarithmic functions.*

Teacher Preparation and Notes

- *This investigation derives the definition of the logarithmic derivative. The students should be familiar with keystrokes for the **limit** command, the **derivative** command, entering the both the natural logarithmic and the general logarithmic functions, drawing a graph, and setting up and displaying a table.*
- *Before starting this activity, students should go to the home screen and select **F6 :Clean Up >2:NewProb**, then press \div . This will clear any stored variables, turn off any functions and plots, and clear the drawing and home screens.*
- *This activity is designed to be student-centered with the teacher acting as a facilitator while students work cooperatively.*

Associated Materials

- Student Worksheet: Sum Rectangles <http://www.ck12.org/flexr/chapter/9731> , scroll down to the second activity.

Problem 1 –The Derivative for

In this problem, students are asked to use the **limit** command (**F3 :Calc >3:limit()**) to find the values. Remind the students to be very careful of their parentheses.

The students should get $\frac{1}{x}$ for the answer to the last problem.

In this portion, the students are asked to use the **derivative** command (**F3 :Calc >1:d(differentiate)**) to find the derivative.

Problem 2 –The Derivative of

Students should notice that both functions have a value of zero for $\ln(1)$ and $\log_2(1)$ but that $\log_2(x)$ is a multiple of $\ln(x)$. They both have about the same shape but $\ln(x)$ is smaller (both positively and negatively) than $\log_2(x)$.

When $\log_4(x)$ is compared to $\ln(x)$, they also intercept at 1 but $\ln(x)$ is larger.

Have the students notice that $\ln(4)$ is listed as $2(\ln(2))$ instead of $\ln(4)$. The answer to the last problem is $\ln(a)$.

Students should graph the functions $y_1 = \ln(x)$, $y_2 = \ln(2) \cdot \log_2(x)$, $y_3 = \ln(3) \cdot \log_3(x)$. They should notice that all three functions yield the same graph. Students can check by graphing each one individually.

Students are asked to find the Generalized Logarithmic Rule for Differentiation. They should find $\frac{dy}{dx} = \frac{\log_a(e)}{x}$. Thus, if $y = \log_a(x)$, then $\frac{dy}{dx} = \frac{1}{(x \ln(a))}$.

Problem 3 –Derivative of Exponential and Logarithmic Functions Using the Chain Rule

Students are asked to identify $u(x)$ and a for each function and then find the derivative by hand or using the **Derivative** command to find the derivative.

Recall: $y = a^u \rightarrow \frac{dy}{dx} = a^u \frac{du}{dx}$ where u depends on x .

- $y = \log_a(u) \rightarrow \frac{dy}{dx} = \frac{1}{(u \ln(a))} \cdot \frac{du}{dx}$ or $\frac{dy}{dx} = \frac{\frac{du}{dx}}{u(\ln(a))}$

$$f(x) = 5^{(x^2)}, u(x) = x^2, a = 5$$

- $f'(x) = 2 \cdot \ln(5) \cdot x \cdot 5^{(x^2)}$

$$g(x) = e^{(x^3+2)}, u(x) = x^3 + 2, a = e$$

- $g'(x) = 3 \cdot x^2 \cdot e^{(x^3+2)}$

$$h(x) = \log_3(x^4 + 7), u(x) = x^4 + 7, a = 3$$

- $h'(x) = \frac{4 \cdot x^3 \log_3(e)}{x^4 + 7}$

$$j(x) = \ln\left(\sqrt{x^6 + 2}\right), u(x) = \sqrt{x^6 + 2}, a = e$$

- $j'(x) = \frac{3 \cdot x^5}{x^6 + 2}$

CHAPTER

8

TE Integration Techniques - TI

Chapter Outline

- 8.1 INTEGRATION BY SUBSTITUTION
 - 8.2 INTEGRATION BY PARTS
 - 8.3 CHARGED UP
-

8.1 Integration by Substitution

This activity is intended to supplement Calculus, Chapter 7, Lesson 1.

ID: 9889

Time required: 45 minutes

Topic: Techniques of Integration

- Use substitutions such as $u = \sqrt{ax+b}$ to compute an integral.
- Use **Integral** (on the Calculus menu) to verify manual calculation of integrals.

Activity Overview

In this activity, we explore methods for computing integrals of functions not in one of the standard forms. The focus here is upon the use of substitution to transform the given integral into a standard form.

Teacher Preparation

This investigation offers opportunities for review and consolidation of key concepts related to methods of substitution and integration of composite functions. Opportunities are provided for skill development and practice of the method of taking integrals of suitable functions. As such, care should be taken to provide ample time for ALL students to engage actively with the requirements of the task, allowing some who may have missed aspect of earlier work the opportunity to build new and deeper understanding.

- This activity can serve to consolidate earlier work on the product rule and methods of integration. It offers a suitable introduction to integration by substitution.
- Begin by reviewing the method of differentiation of composite functions (the “chain rule”) and the methods of integration of the standard function forms.
- This activity requires the use of CAS technology for students to check their answers.

Classroom Management

- This activity is designed to be **teacher-led**. Have the students work in pairs. Use the following pages to present the material to the class and encourage discussion. Although the majority of the ideas and concepts are presented in **this** document, be sure to cover all the material necessary for students’ total comprehension.
- The students will need to be able to compute the derivative and the indefinite integral on their own.
- Answers should be completed on the student worksheet.
- Before starting this activity, students should go to the home screen and select **F6 :Clean Up >2:NewProb**, then press **ENTER**. This will clear any stored variables, turn off any functions and plots, and clear the drawing and home screens.

Associated Materials

- Student Worksheet: Integration by Substitution <http://www.ck12.org/flexr/chapter/9732>

Problem1 –Introduction

Step 1: Begin with discussion and review of both the chain rule for differentiation of composite functions and of the integrals of standard function form.

Ensure that students are comfortable with these and then challenge them to consider more difficult forms –in this case, composite functions of the form $y = f(g(x))$ which may be suitable for integration by substitution methods.

Step 2: Students are to use substitution to integrate $\int \sqrt{2x+3} \, dx$. They can use the table on the worksheet to help guide them through the steps.

Students should use the selection of u to compute du and rewrite the function in terms of u . Then they integrate it with respect to u and rewrite the evaluation of the integral back in the original variable.

Note: Some students may not realize that they need to have the $\frac{1}{2}$ in front of the integral. To help them see where it comes from, have them solve du for dx .

In addition, they should use the **integral** command (**Home > F3 :Calc >2:Integral**) or (**Home > 2nd [7]**) to check their answer. Students should be aware that integrals evaluated by CAS generally do not include the constant term.

Step 3: The integral for $\sin(x)\cos(x)dx$ is developed three ways. First, students let $u = \sin(x)$ and then let $u = \cos(x)$. Students should see that the results are different, but only a constant apart.

The third way students find the integral is by transforming $\sin(x)\cos(x)dx$ into $\frac{1}{2}\sin(2x)$ using the Double Angle formula, letting $u = \sin(2x)$.

Students can check the answer with their calculator.

Problem2 –Common Feature

Step 4: Students are to work through three problems using substitution to integrate. First, have them discuss what they should use for u in each of the problems. If students are having trouble with the steps, have them create a table like the one on the worksheet.

Step 5: After working through the problems, students are challenged to identify the common feature: that each of the given functions in some way includes the derivative of the function substituted. It is critical for students to understand that this method will not work for all functions, but only for certain well chosen forms.

Extension

The challenge to the students in these problems is to use trig identities to rewrite the integral so that they have a substitution format.

$$\begin{aligned}\tan(x) &= \frac{\sin(x)}{\cos(x)} \\ \cos(x)^3 &= \cos(x)\cos(x)^2 = \cos(x)(1 - \sin(x)^2)\end{aligned}$$

Solutions

1. $u = 2x + 3; du = 2dx; \int \sqrt{2x+3} dx = \frac{1}{2} \int u^{\frac{1}{2}} du = \frac{1}{3} u^{\frac{3}{2}} + C = \frac{1}{3} (2x+3)^{\frac{3}{2}} + C$
2. $u = \sin(x); du = \cos(x)dx; \int \sin(x) \cos(x)dx = \int u du = \frac{1}{2} u^2 + C = \frac{1}{2} \sin(x)^2 + C$
3. $u = \cos(x); du = -\sin(x)dx; \int \sin(x) \cos(x)dx = -\int u du = -\frac{1}{2} u^2 + C = -\frac{1}{2} \cos(x)^2 + C$
4. $u = 2x; du = 2dx; \int \frac{1}{2} \sin(2x)dx = \frac{1}{4} \int \sin(u)du = -\frac{1}{4} \cos(u) + C = -\frac{1}{4} \cos(2x) + C$
- 5.

$$u = x^2 + 2x + 3; du = (2x + 2)dx;$$

$$\int \frac{x+1}{x^2+2x+3} dx = \frac{1}{2} \int \frac{1}{u} du = \frac{1}{2} \ln(u) + C = \frac{1}{2} \ln(x^2 + 2x + 3) + C$$

6. $u = \cos(x); du = -\sin(x)dx; \int \sin(x) e^{\cos(x)} dx = -\int e^u du = -e^u + C = -e^{\cos(x)} + C$
7. $u = 4x^2 + 1; du = 8x dx; \int \frac{x}{4x^2+1} dx = \frac{1}{8} \int \frac{1}{u} du = \frac{1}{8} \ln(u) + C = \frac{1}{8} \ln(4x^2 + 1) + C$
8. Each contains the derivative of the substitution element: if u is the substitution, then $\frac{du}{dx}$ exists as part of the expression, or at least a constant multiple of it.
9. $u = \cos(x); du = -\sin(x); \int \frac{\sin(x)}{\cos(x)} dx = -\int \frac{1}{u} du = -\ln(u) + C = -\ln(\cos(x)) + C$
10. $u = \sin(x); du = \cos(x)dx;$

$$\int \cos(x)(1 - \sin(x)^2)dx = \int (1 - u^2)du = u - \frac{u^3}{3} + C = \sin(x) - \frac{\sin(x)^3}{3} + C$$

8.2 Integration by Parts

This activity is intended to supplement Calculus, Chapter 7, Lesson 2.

ID: 9850

Time required: 45 minutes

Topic: Techniques of Integration

- *Derive the formula for integration by parts and use it to compute integrals.*
- *Use successive applications of integration by parts to compute integrals.*
- *Use **Integral** (on the Calculus menu) to verify manual calculation of integrals.*

Activity Overview

In previous activities, students have explored the differential calculus through investigations of the methods of first principles, the product and quotient rules. In this activity the product rule becomes the basis for an integration method for more difficult integrals.

Teacher Preparation

This investigation offers opportunities for review and consolidation of earlier key concepts related to standard integrals and the product rule for differentiation as well as the mastery of the method of integration by parts. As such, care should be taken to provide ample time for ALL students to engage actively with the requirements of the task, allowing some who may have missed aspect of earlier work the opportunity to build new and deeper understanding.

- *This activity can serve to consolidate earlier work on the product rule and methods of integration. It offers a suitable introduction to the method of integration by parts.*
- *Begin by discussing approaches to more difficult integrals and review the methods used for differentiation. It is advisable that students have some experience with substitution methods for integration before attempting this activity.*
- *The screenshots on pages 1–2 demonstrate expected student results.*
- *To download the student worksheet, go to education.ti.com/exchange and enter “9850” in the quick search box.*

Classroom Management

- *This activity is designed to be **teacher-led**. Have the students work in pairs. Use the following pages to present the material to the class and encourage discussion. Although the majority of the ideas and concepts are presented in **this** document, be sure to cover all the material necessary for students’ total comprehension.*
- *The students will need to be able to compute the derivative and the indefinite integral on their own.*
- *Answers should be completed on the student worksheet.*
- *Before starting this activity, students should go to the home screen and select **F6 :Clean Up >2:NewProb**, then press \div . This will clear any stored variables, turn off any functions and plots, and clear the drawing and home screens.*

Associated Materials

- Student Worksheet: Integration by Parts <http://www.ck12.org/flexr/chapter/9732> , scroll down to the second activity.

Integration by Parts

Step 1: Begin with discussion concerning integration methods and review standard integrals. Some review of substitution methods of integration, such as $\int \cos(2x+1)dx$, is recommended.

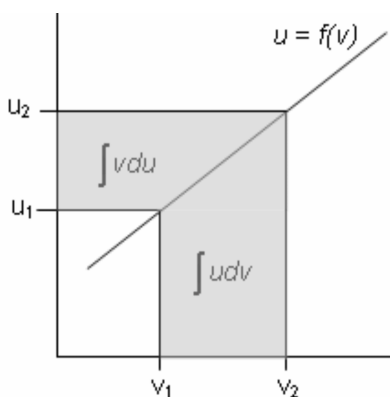
Students should be aware that not all functions can be integrated symbolically.

Step 2: Students are to complete exercises 1 and 2 on the worksheet. They will use the product rule on the function $f(x) = \sin(x) \ln(x)$. Students should be given opportunities to review and consolidate their skills and understandings related to both product rule and the process of integration (particularly drawing distinction between the integral of a function and the area under the curve of the graph of that function).

Step 3: The idea of taking the integral of a derivative may well be new to students and should be handled carefully, ensuring that the errors and misunderstandings are exposed if possible. Students should be aware that integrals evaluated by CAS generally do not include the constant term.

Step 4: Students are to find the integral of both sides of the product rule. Integrating the product rule and rearranging the parts provides an expression which proves to be the basis for our method of integration by parts. However, it is recommended that this process is not assumed to be valid, but that students be given the opportunity to engage with it, to discuss and to challenge.

Step 5: Students are to complete exercise 6 on the worksheet. A graphical approach can help students appreciate the significance of each of the components parts of integration by parts statement: The areas under the curve **with respect to u** and **with respect to v** are key concepts here. Interestingly, it is the product u^*v which is most difficult to draw from this scenario. It represents a shorthand for the difference product, $u_2 \cdot v_2 - u_1 \cdot v_1$. This is shown on the diagram as the L -shaped area between the curve and the axes, between the given limits.



Step 6: Drawing together these observations, students should be assisted to see the significance of this method. Integration by parts allows students to find the integral of a function that is a product by taking the derivative of one component and the integral of the other.

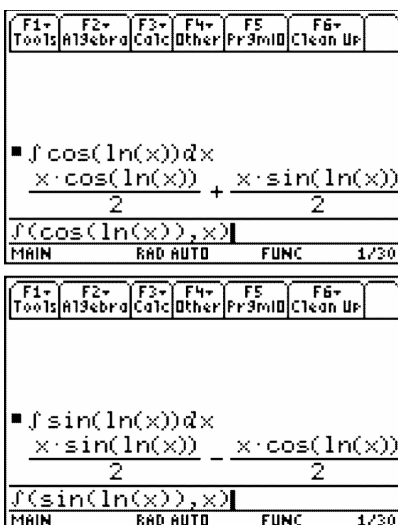
Step 7: Students are to complete exercise 7 by integrating $\ln(x)$ using integration by parts. They can use the home screen on the calculator to check their work.

The challenge for students when using integration by parts is which part to make u and which part to make dv . Whether this choice makes a difference is an extension exercise.

Even though this function does not appear to be a product we can treat it as one.

Step 8: Students are to complete the second part of finding the integral of $\sin(\ln(x))$. They will need to use integration by parts to find the integral of $\cos(\ln(x))$ and then substitute their result into the result for $\sin(\ln(x))$. The integral check is the screen to the right.

This exercise with its companion below shows the cyclical work for the trig functions.



Step 10: Students are to use integration by parts to find the integrals of the four functions on their worksheet.

Reduction rules may also be developed in the way 10 b is worked. This method leads to a method for integrating trig functions of higher powers.

Step 11: The extension challenges are added in the student solutions.

Solutions

- $\frac{d}{dx}(u \cdot v) = u \frac{dv}{dx} + v \frac{du}{dx}$
- $\frac{d}{dx}(\sin(x) \cdot \ln(x)) = \sin(x) \cdot \frac{1}{x} + \ln(x) \cdot \cos(x)$
- No; While $\frac{d}{dx}(\int f(x) dx) = f(x)$ is true, $\int \frac{d}{dx}(f(x)) dx = f(x) + C$
- $\int \left(\frac{d}{dx}(u(x) \cdot v(x)) \right) dx = u(x) \cdot v(x) + C$
- $v(x) \cdot \int u(x) dx + u(x) \cdot \int v(x) dx$
- The area between the curve $u = f(v)$ and the v -axis (between the specified limits, i.e. $\int_{v_1}^{v_2} u \cdot dv$) can be found by taking the area of the large rectangle ($u_2 \cdot v_2$) minus the smaller rectangle ($u_1 \cdot v_1$), and then subtracting the area of the region between the curve and u -axis, between the specified limits $\left(\int_{u_1}^{u_2} v \cdot du \right)$.
- $\int (\ln(x) \cdot 1 dx)$ using Integration by Parts

$$\begin{aligned} u(x) &= \ln(x) & du &= \frac{1}{x} dx \\ dv &= 1 dx & v &= x \end{aligned}$$

$$\begin{aligned}\int \ln(x) \cdot 1 dx &= \ln(x) \cdot x - \int x \cdot \frac{1}{x} dx \\ &= x \cdot \ln(x) - x\end{aligned}$$

$$8. \int \cos(\ln(x)) dx = x \cdot \cos(\ln(x)) + \int \sin(\ln(x)) dx$$

9.

$$\begin{aligned}\int \sin(\ln(x)) \cdot 1 dx &= x \sin(\ln(x)) - 2 \left[x \cos(\ln(x)) + \int \sin(\ln(x)) dx \right] \\ &= \frac{x}{2} [\sin(\ln(x)) - \cos(\ln(x))]\end{aligned}$$

10. a) $\int \tan^{-1}(x) \cdot 1 dx$ using Integration by Parts

$$\begin{aligned}u(x) &= \tan^{-1}(x) & du &= \frac{1}{x^2 + 1} dx \\ dv &= 1 dx & v(x) &= x\end{aligned}$$

$$\begin{aligned}\int \tan^{-1}(x) \cdot 1 dx &= \tan^{-1}(x) \cdot x - \int x \cdot \frac{1}{x^2 + 1} dx \\ &= x \tan^{-1}(x) - \frac{1}{2} \ln(x^2 + 1) + C\end{aligned}$$

b) $\int x^2 \cdot e^x dx$ using Integration by Parts

$$\begin{aligned}u(x) &= x^2 & du &= 2x \\ dv &= e^x dx & v(x) &= e^x\end{aligned}$$

$$\int x^2 \cdot e^x dx = x^2 \cdot e^x - \int e^x \cdot 2x dx$$

Repeat Integration by Parts for $\int e^x \cdot 2x dx$

$$\begin{aligned}u(x) &= 2x & du &= 2 dx \\ dv &= e^x dx & v(x) &= e^x\end{aligned}$$

$$\int e^x \cdot 2x dx = 2x \cdot e^x - 2 \int e^x dx = 2x \cdot e^x - 2e^x$$

Hence, $\int x^2 \cdot e^x dx = x^2 \cdot e^x - 2x \cdot e^x + 2e^x = e^x(x^2 - 2x + 2)$

c) $\int x \tan^{-1}(x) dx$ using Integration by Parts

$$\begin{aligned}
 u(x) &= \tan^{-1}(x) & du &= \frac{1}{x^2+1} dx \\
 dv &= x \, dx & v(x) &= \frac{x^2}{2}
 \end{aligned}$$

$$\begin{aligned}
 \int x \tan^{-1}(x) &= \tan^{-1}(x) \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{1}{x^2+1} dx \\
 &= \frac{x^2 \tan^{-1}(x)}{2} - \frac{1}{2} \int \frac{x^2+1}{x^2+1} - \frac{1}{x^2+1} dx \\
 &= \frac{x^2 \tan^{-1}(x)}{2} - \frac{1}{2} (x - \tan^{-1}(x)) \\
 &= \frac{1}{2} ((x^2+1) \tan^{-1}(x) - x)
 \end{aligned}$$

d) $\int x \cos(2x+1) dx$ using Integration by Parts

$$\begin{aligned}
 u(x) &= x & du &= 1 \, dx \\
 dv &= \cos(2x+1) \, dx & v(x) &= \frac{\sin(2x+1)}{2} \\
 \int x \cos(2x+1) &= x \cdot \frac{\sin(2x+1)}{2} - \int \frac{\sin(2x+1)}{2} \cdot 1 \, dx \\
 &= \frac{x \sin(2x+1)}{2} + \frac{\cos(2x+1)}{4} \\
 &= \frac{1}{4} (2x \sin(2x+1) + \cos(2x+1))
 \end{aligned}$$

11. Often it does not matter, if both functions are readily differentiable and integrable, but in cases where one may not be reduced by the process, it can make a difference.

12. No, since the quotient rule formula does not allow us to easily rearrange the parts –the denominator v^2 causes problems.

8.3 Charged Up

This activity is intended to supplement Calculus, Chapter 7, Lesson 7.

ID: 12368

Time Required: 20 minutes

Activity Overview

*In this activity, students will be asked some exam-like questions to introduce separable differential equations. Students will use the **deSolve** command to find general and particular solutions to differential equations. They will also graphically view the family of particular solutions to a differential equation.*

Topic: Differential Equations

- Differential equations solved algebraically. Solutions shown graphically.
- Use TI-89 to find solutions to DEs.

Teacher Preparation and Notes

- The syntax for **deSolve** is **deSolve**($y' = f(x,y), x, y$) where x is the independent and y is the dependent variable. The **deSolve** command can be found in the **HOME** screen by pressing F3 or F4 and selecting **C: deSolve**. This activity will help students see the application of DEs, the process of solving separable DEs, and how to use the TI-89 to check or find the solution to DEs.
- Students will write their responses directly on the accompanying handout.
- To download the calculator file, go to <http://www.education.ti.com/calculators/downloads/US/Activities/Detail?id=12368> and select Diffq1 and Diffq2.

Associated Materials

- Student Worksheet: Charged Up <http://www.ck12.org/flexr/chapter/9732> , scroll down to the third activity.
- Diffq1.89d
- Diffq2.89i

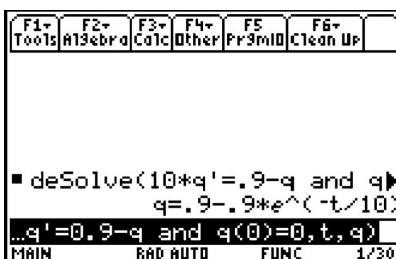
Part 1 –Separable Differential Equation Introduced

For Questions 1 and 2, students are asked exam like questions about differential equations. These foundational (practically review questions) can serve as an introduction to separable differential equations. Teachers can use question 2 to formatively assess how well students are making the connection to their previous knowledge. If students appear to be struggling, teachers can use the opportunity to state the following steps: STEP 1 separate variable, STEP 2 integrate both sides, STEP 3 apply initial conditions to solve for the constant of integration and find a particular solution, STEP 4 answer the question.

DEs are applicable for all sorts of physical phenomena, including radioactive decay, economics, biology, chemistry, population growth and electric circuits with capacitors.

The steps for solving a separable DE are outlined and reinforced. Some students may need help with the algebra involved in the steps.

The **deSolve** command is used to check the solution. The syntax of **deSolve** is explained so that students can use this as a tool to explore several other DEs.



Student Solutions

- $\frac{dq}{dt} = kq$
- $\int dy = \int \sin(x) \cos^2(x) dx \Rightarrow y = -\frac{1}{3} \cos^3(x) + C$ and since $\cos\left(\frac{\pi}{2}\right) = 0, y(0) = -\frac{1}{3}$.
- $\frac{dq}{0.9-q} = \frac{1}{10} dt \Rightarrow -\ln|0.9-q| = \frac{1}{10}t + C \Rightarrow \ln|0.9-q| = -\frac{1}{10}t + C$
- $|0.9-q| = e^{-\frac{1}{10}t+C} = e^{-\frac{1}{10}t}e^C$. Let $C_1 = \pm e^C, 0.9-q = C_1e^{-\frac{1}{10}t}$ and with $q = 0$ when $t = 0, C_1 = 0.9$

$$q(t) = 0.9 \left(1 - e^{-\frac{1}{10}t}\right)$$

- They are equivalent. In the solution using **deSolve**, the 0.9 was distributed

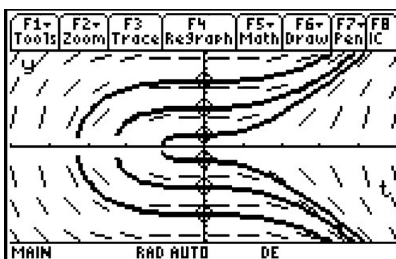
$$6. y = c \cdot x$$

If $y(1) = 1, c = 1$, so the particular solution is $y = x$.

Part 2 –Homework/Extension –Practice with deSolve and Exploring DEs

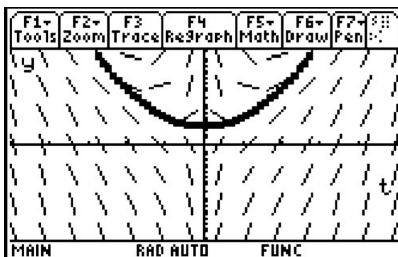
Students first use **deSolve** to find the general solution to four DEs. The fourth situation can also be viewed graphically. To view the graph of the family of functions, students should enter **RclGDB diffq1** on the **HOME** screen

On the worksheet students will write the solution to each DE. You may wish for students to algebraically show their work and use **deSolve** check their answers.



A non-separable DE is solved with **deSolve**. The particular solution can also be viewed graphically by entering **RclPic diffq2** on the **HOME** screen. The slope field and the particular solution are shown on the screen.

Finally, **deSolve** is used to find the particular solution for three DEs with initial conditions. Students are asked to solve for y . With the TI-89, this can be done with **solve()**. Again, students may show their work and use **deSolve** to check their answers.



Another extension/exploration activity would be to have students come up with their own DE and find the general solution. Have them discuss solutions that surprised them.

Student Solutions

1. $y = Ce^{kx}$
2. $y^2 = x^2 + C$
3. $y^3 = 3x^2 + C \rightarrow y = (3x^2 + C)^{\frac{1}{3}}$
4. $y^2 = 2x^3 + C$
5. It looks like a parabola because $y = x^2 + \frac{c}{x} + 2$ is $y = x^2 + 2$ when $c = 0$.
6. $1 - \frac{1}{y} = \frac{x^2}{2} \Rightarrow y = \frac{-2}{x^2 - 2}$
7. $\tan^{-1}(y) + \frac{\pi}{4} = x \rightarrow y = \tan\left(x - \frac{\pi}{4}\right)$
8. $y = e^{7x}$

CHAPTER

9

TE Infinite Series - TI

Chapter Outline

- 9.1 EXPLORING GEOMETRIC SEQUENCES
 - 9.2 INFINITE GEOMETRIC SERIES
 - 9.3 MR. TAYLOR, I PRESUME?
 - 9.4 AP* CALCULUS EXAM PREP
-

9.1 Exploring Geometric Sequences

This activity is intended to supplement Calculus, Chapter 8, Lesson 1.

ID: 10992

Time required: 20 minutes

Activity Overview

In this activity, students will explore geometric series. They will consider the effect of the value for the common ratio and first term using the TRANSFRM APP for the TI-84. Students will graphically analyze geometric series using graphs.

Topic: Sequences and Series

- *Explore geometric sequences*

Teacher Preparation and Notes

- *This activity serves as a nice introduction to geometric series. Students will need the TRANSFRM APP on their TI-84. The accompanying worksheet can add depth to the questions and helpful key press explanation.*
- *An extension would be to solve infinite geometric series that converge using sigma notation and the limit of the partial sum formula.*
- *Students should make sure to turn off all Stat Plots before beginning the activity.*
- *To exit the Transformation App, go to the App menu, press enter on Transfrm and then select Uninstall.*
- *To download the Transformation Graphing Application, go to <http://www.education.ti.com/calculators/downloads/US/Activities/Detail?id=10992> and select Transformation Graphing from the Applications header.*

Associated Materials

- Student Worksheet: Exploring Geometric Sequences <http://www.ck12.org/flexr/chapter/9733>
- Transformation Graphing Application

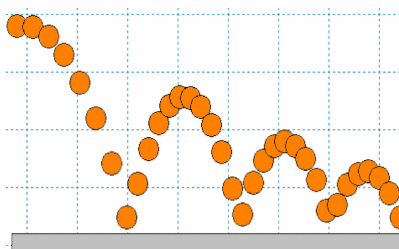
Example of Geometric Sequence

Students are shown the path of a ball that is bouncing. Show that the common ratio of the heights is approximately the same.

$$\frac{2.8}{4.0} = 0.7$$

$$\frac{2.0}{2.8} \approx 0.71$$

$$\frac{1.4}{2.0} = 0.7$$



Changing the Common Ratio

For the first part of this activity, students explore geometric sequences graphically by varying the value of r , the common ratio. Pressing the right or left arrows will change the value of r and updates the graph each time. Note that students will not see a graph for A values less than zero.

They should see that r -values between 0 and 1 could model the heights of the ball in the example.

Inquiry questions

- Why do you think the r -value is called the common ratio? [Answer: The ratios of consecutive terms are the same; They have that in common.]
- What would happen if you added all the terms of this sequence? For what common ratio conditions do you think the sum will diverge (get larger, and not converge to some number)?

```

Plot1 Plot2 Plot3
M1Y1=4*A^(X-1)
M1Y2=
M1Y3=
M1Y4=
M1Y5=
M1Y6=
M1Y7=

```

```

WINDOW SETTINGS
X1 > X2 > X3 >>
HE0.7
Step=.1

```

Changing the Initial Value and the Common Ratio

Students will now change the equation in the Transform App to explore the changes in the graph of the general series, $a_n = a_1 \cdot r^{n-1}$. Students use the up and down arrow to move from one variable to another and use the left and right arrows to change the value of the variable. They should be sure to try negative and positive values for B for various values of A . ($Y1 = B \cdot A^{(X-1)}$)

Inquiry question

- Which variable seems to have a more profound effect on the sequence? Explain.

Extension –Deriving and Applying the Partial Sum Formula

On the worksheet, students are shown the derivation of the formula for the sum of a finite geometric series. You may need to explain in detail the substitution in the third line. For example, $a_2 = r \cdot a_1$, so then $a_3 = r \cdot a_2 = r \cdot (r \cdot a_1) = r^2 \cdot a_1$. In the fourth line, r is multiplied by both sides, changing r^{n-1} to r^n .

Students can apply the formula to find the sum of the series given on the worksheet.

Inquiry questions:

- If you cut a piece of paper in half, then cut one of the halves in half and repeated the process, what is the sum of all these fractional pieces? [Answer: one whole]
- However, if you gave your teacher one penny on the first day of class, 2 on the next, then 4, 8, 16, 32, ... would this number converge?
- How would the formula for a finite geometric sequence look if r is a fractional number and n is increasing?

9.2 Infinite Geometric Series

This activity is intended to supplement Calculus, Chapter 8, Lesson 2.

ID: 11065

Time required: 45 minutes

Activity Overview

In this activity, students will explore infinite geometric series. They will consider the effect of the value for the common ratio and determine whether an infinite geometric series converges or diverges. They will also consider the derivation of the sum of a convergent infinite geometric series and use it to solve several problems.

Topic: Sequences Series

- Explore geometric sequences
- Sum a geometric series
- Convergence of an infinite geometric series

Teacher Preparation and Notes

- This activity serves as an introduction to infinite geometric series. Students will need to have previously learned about finite geometric series.
- Before beginning the activity, students need to clear all lists and turn off all plots and equations. To clear all lists, press 2nd **LIST**, scroll down to **ClrAllLists** and press **ENTER**

Associated Materials

- Student Worksheet: Infinite Geometric Series <http://www.ck12.org/flexr/chapter/9733> , scroll down to the second activity.

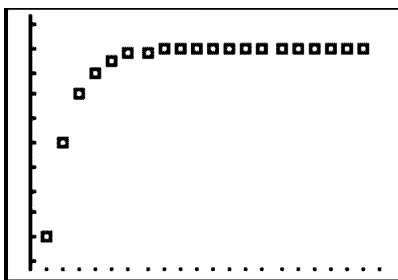
Problem 1 –Investigating Infinite Geometric Series

Students will explore what happens when the common ratio changes for an infinite geometric series. For each value of r , students will create lists $L2$ and $L3$ and then scroll through $L3$ to determine if the series converges or diverges.

As an extension, students could change the initial value of the sequence by changing the number 200 in the formula for $L2$ and see if it affects the convergence or divergence of the series.

If students determine that a series converges, then they are to create and view the scatter plot. The necessary settings for **Plot1** are shown on the student worksheet.

To change the window, students can press **#** and select **ZoomStat** or manually adjust it by pressing **ZOOM**.



1.

TABLE 9.1:

r	-2	-0.5	-0.25	0.25	0.5	2
Converges or Diverges	Diverges	Converges	Converges	Converges	Converges	Diverges
		133.33	160	266.667	400	

2. $|r| < 1$

3. There is a horizontal asymptote at the point of convergence.

Problem 2 –Deriving a Formula for the Sum of a Convergent Infinite Geometric Series

Students are to use the Home screen to determine the values of r^n when $r = 0.7$. They should see that as n gets very large r^n becomes zero when $|r| < 1$.

Students are given the formula for the sum of a *finite* geometric series. With the information found on the worksheet, they can determine the formula for the sum of an *infinite* geometric series using substitution.

4. Note that even though the calculator says r^{1000} and r^{10000} both equal zero, they are actually very small numbers and are *approximately* zero.

TABLE 9.2:

n	10	100	1000	10000
$r^n = 0.7^n$	0.028248	3.23 E-16	0	0

5. $s_n = \frac{a_1(1-r)}{1-r} = \frac{a_1}{1-r}$

Problem 3 –Apply what was learned

In this problem, students are given a scenario relating to drug prescriptions and dosages. Students need to use the formulas shown in the previous problem to answer the questions.

They may get caught up on the first question. Explain to students that if 15% of the drug leaves the body every hour, then that means that 85% percent is still in the body.

6. a. $0.40 = 1 - 0.15(4)$

b. $240 + 240(0.4) = 336 \text{ mg}$

c.

TABLE 9.3:

Hours	0 (1st dosage)	4 (2 nd dosage)	8 (3 rd dosage)	12	16
Amount in the Body	240	336	374.4	389.76	395.904

d. This is the 7th dosage, so $S_7 = \frac{240(1-.4^7)}{1-.4} = 399.34464$.

e. This is the 19th dosage, so $S_{19} = \frac{240(1-.4^{19})}{1-.4} = 399.999989$.

f. $S_t = \frac{240(1-.4^t)}{1-.4}$

g. No, since $S = \frac{240}{1-.4} = 400$, you will not reach the minimum lethal dosage.

f. Yes, since he/she waits 2 hours, only 30% of the drug is out of his/her system, so 70% remains. This is the common ratio $r = 0.7$ and $S = \frac{240}{1-.7} = 800$

9.3 Mr. Taylor, I Presume?

This activity is intended to supplement Calculus, Chapter 8, Lesson 7.

ID: 10257

Time required: 45 minutes

Activity Overview

In this activity, students will use the TI-89 to explore Taylor polynomials graphically and analytically. They will also graphically determine the interval where the Taylor polynomial closely approximates the function it models.

Topic: Series and Taylor Polynomials

- *Display in a spreadsheet the first few terms of a Taylor series approximation to $f(x)$ for a given value of x and compare the value of the Taylor approximation with the value of $f(x)$.*
- *Use Taylor Polynomial (on the Calculus menu) to verify the manual computation of a Taylor Series.*
- *Graph a function and its Taylor polynomials of various degrees to show their convergence to the function.*

Teacher Preparation and Notes

- *Students should be familiar with basic differentiation prior to the beginning of this activity.*
- *Students should be able to graph functions, and set up tables on their calculator. They should also be able to use commands and set up tables on their own.*
- *This activity is designed to be **teacher-led**.*
- *Before starting this activity, students should go to the Home screen and select F6 :**Clean Up >2:NewProb**, then press **ENTER**. This will clear any stored variables, turn off any functions and plots, and clear the drawing and home screens.*

Associated Materials

- Student Worksheet: Mr. Taylor, I Presume? <http://www.ck12.org/flexr/chapter/9733> , scroll down to the third activity.

Introduction

This activity begins with students finding a polynomial given only the value of its derivatives at a specific x -value. This is at the heart of finding Taylor polynomials. If students struggle understanding that this method works, it may be necessary to reverse the process and have the students find the derivatives and observe that their values at the given x -value are the given values of the derivative.

At the conclusion of the introductory problem, the Taylor polynomial form is found when centered at zero. At this time, place the general form of the Taylor polynomial when $x = 0$ on the board.

$$P_n(x) = \frac{f^{(0)}(a)}{0!}(x-a)^0 + \frac{f'(a)}{1!}(x-a)^1 + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Explain that when $a = 0$, this matches the form $P(x) = a_0 + a_1x + a_2x^2 + a_3x^3$

Taylor polynomial centered at zero

Students are asked to find the 4th degree Taylor polynomial that approximates $f(x) = \ln(x+5)$ at $x = 0$. The derivatives of the function are:

$$\begin{aligned} f(0) &= \ln(5) \approx 1.609 \\ f'(x) &= \frac{1}{x+5} \rightarrow f'(0) = \frac{1}{5} \\ f''(x) &= \frac{-1}{(x+5)^2} \rightarrow f''(0) = -\frac{1}{25} \\ f'''(x) &= \frac{2}{(x+5)^3} \rightarrow f'''(0) = \frac{2}{125} \\ f^{(4)}(x) &= \frac{-6}{(x+5)^4} \rightarrow f^{(4)}(0) = -\frac{6}{625} \end{aligned}$$

F1 Tools	F2 Algebra	F3 Calc	F4 Other	F5 Pr3mID	F6 Clean Up
$\blacksquare \text{taylor}(\ln(x+5), x, 4, 0)$ $-\frac{x^4}{2500} + \frac{x^3}{375} - \frac{x^2}{50} + \frac{x}{5} + \ln(5)$					
$\text{taylor}(\ln(x+5), x, 4, 0)$					
MAIN RAD AUTO FUNC 1/20					

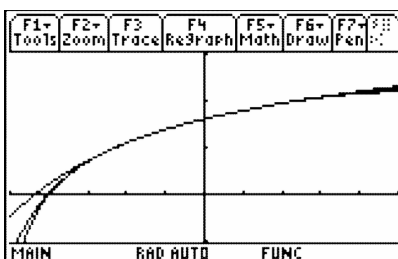
As the students compare the values of the Taylor polynomial with the values of the original function, they will notice that the values are closest at the center (the x -value where the derivatives were found) and become farther apart the further the x -values are from the center (the graph demonstrates this even further). Point out to students that not all Taylor polynomials have such a large interval as this one.

The table at the right has the 4th power and 10th power Taylor polynomials as well as the function.

F1- Tools	F2- Setup	F3- Calc	F4- Other	F5- Pr3mID	F6- Clean Up
x	y1	y2	y3		
-2.	1.0986	1.1017	1.0986		
-1.	1.3863	1.3864	1.3863		
0.	1.6094	1.6094	1.6094		
1.	1.7918	1.7917	1.7918		
2.	1.9459	1.9444	1.9459		
x=-2.					
MAIN		RAD AUTO		FUNC	

Students will also have a chance to explore different degrees of a Taylor polynomial. It would be best if you choose three larger values ahead of time. Have the students use the **Taylor** command to find the respective polynomials

and graph the function with each of the larger power polynomials in turn. If time permits, have a few students find the 10th degree Taylor polynomial and graph it on the same set of axes as the 4th degree Taylor and the original function. This will demonstrate that a Taylor polynomial will only approximate the function over a given interval, no matter how large the Taylor Polynomial is. It is important for students to know that a Taylor polynomial centered at 0 is known as a Maclaurin polynomial.



Taylor polynomial not centered at zero

When students work the problem of where the Taylor polynomial is not centered at zero, they will observe that the polynomial follows the original function for a much smaller interval. Remind students that they need to use $(x - a)^n$ instead of x^n .

The derivatives for this problem are found below.

TABLE 9.4:

	Derivative	Value at $x = 1$
$f(x)$	$\frac{1}{2-x}$	1
$f'(x)$	$-(2-x)^{-2}(-1) = \frac{1}{(2-x)^2}$	1
$f''(x)$	$-2(2-x)^{-3}(-1) = \frac{2}{(2-x)^3}$	2
$f'''(x)$	$2(-3)(2-x)^{-4}(-1) = \frac{6}{(2-x)^4}$	6
$f^{(4)}(x)$	$6(-4)(2-x)^{-5}(-1) = \frac{24}{(2-x)^5}$	24

The Taylor polynomial can be:

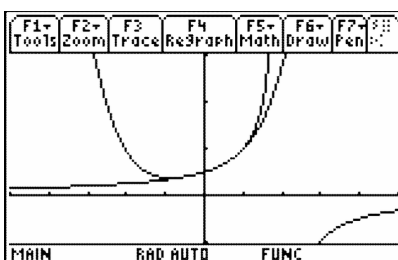
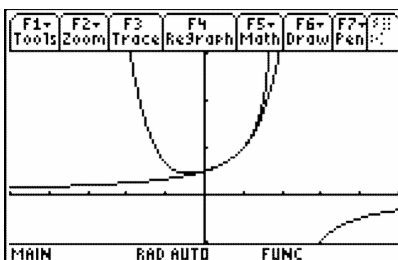
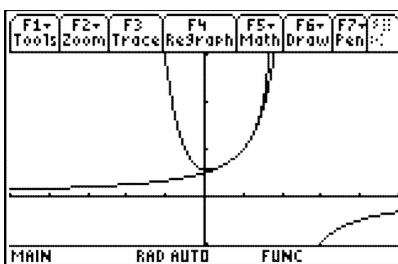
$$P(x) = \frac{1}{0!}(x-1)^0 + \frac{1}{1!}(x-1)^1 + \frac{2}{2!}(x-1)^2 + \frac{6}{3!}(x-1)^3 + \frac{24}{4!}(x-1)^4$$

or

$$P(x) = 1 + (x-1) + (x-1)^2 + (x-1)^3 + (x-1)^4$$

Pick three values from 0 to 1 (for instance 0.8, 0.5, 0.2) and find the 4th degree Taylor polynomial using the **Taylor** command. Graph each Taylor polynomial and the original function the same set of axes separately. Students should notice that the interval increases as the center gets closer to zero.

See the graphs below.



9.4 AP* Calculus Exam Prep

This activity is an end-of-the-year review for the AP Calculus exam.

ID: 12623

Time Required: 20 minutes

Activity Overview

Students will prepare for the AP Calculus exam. Students will do exercises and learn practice tips. Students will review and gain skills in such topics as graphing, finding intersections, finding zeros, evaluating derivatives at a point, and evaluating definite integrals. Examples of non-calculator multiple-choice questions are also provided.

Topic: AP Exam-Type Questions

- *Using a graph effectively to find zeros and intersection points.*
- *Using calculator features to help answer multiple-choice questions.*

Teacher Preparation and Notes

- *This activity focuses on AB-type questions, since the BC exam shares many of the AB questions.*
- *After completing the activity, students should be more successful in answering many of the multiple-choice questions, especially from 2003. From the calculator-active questions, have students try 2003AB76, 81, 83, 84, 92.*
- *For practice AP exam questions from the Princeton Review, go to http://education.ti.com/educationportal/site/US/nonProductMulti/test_prep.html?bid=3.*

Associated Materials

- Student Worksheet: AP Calculus Exam Prep <http://www.ck12.org/flexr/chapter/9733>, scroll down to the fourth activity.

** AP, College Board, and SAT are registered trademarks of the College Board, which was not involved in the production of and does not endorse this product.*

Part 1 –Calculator Active Exam Tips

Students will read and answer questions in their worksheet to learn and review tips for taking the AP exam. These questions are similar to some of the calculator active questions on the 2003 exam.

Although students should use the graphing calculator throughout the year that they will use on the exam, this activity was written at an introductory level. Key presses are provided on the student worksheet. Students are allowed to bring two graphing calculators, so some may consider bringing a TI-89 graphing calculator or TI-Nspire CAS handheld in addition to their normal calculator.

Student Solutions

1. About 15 minutes should be spent for each free response question (45 minutes for 3 calculator active, then 45 minutes for no calculator permitted). Section 1 Part A students should spend just under 2 minutes for each non calculator permitted question. Section 1 Part B students should spend just under 3 minutes for the 17 calculator active questions.
2. $\frac{d}{dt}(3t + 6.5 \cos(2t))|_{t=2} = -5.497$
3. $-1.622 < x < -1.290$; This is where the derivative of the integral is positive. Therefore this is where the function $h(x)$ is increasing.
4. $\int_{xc}^3 (3 \ln(x) - \frac{1}{x}) dx = 2.928$, where xc is the x -coordinate of the intersection and $xc = 1.29386$

Question 4 pertains to free-response questions. This point of intersection will be used for multiple parts of a free-response question.

Students should practice more calculator active questions. Old exams are available through College Board.

Part 2 –AB exam practice non-calculator type questions

This section allows students to practice and build their confidence with *AB* non-calculator questions that are similar to the 2003 exam.

Students are reminded to be well rested for this marathon of the mind. They should go into the exam room with confidence and peace, planning to do the best they can based on their training and ability.

The solutions below include a rule, method, or other specific knowledge that was used to solve the problem.

Discussion Question:

- Ask students to reflect on what they have learned in the first part of the activity. What new skills or important tips were they reminded of?

Student Solutions

1. D –this is a simple chain rule question. Make sure students don't forget to multiply by the derivative of what's 'inside.'
2. E –Fundamental Theorem of Calculus with Chain Rule and remembering that $e^0 = 1$.
3. B –Implicit differentiation.

$$\sec^2(x) = e^y \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{\sec^2(x)}{e^y} = \frac{\sec^2(x)}{\tan(x)} = \frac{1}{\sin(x) \cos(x)}$$

4. C –Quotient Rule
5. D –Fundamental Theorem of Calculus and knowledge of basic trigonometric values
6. D –limit of polynomials of same degree (focus on the leading coefficient). The other terms fade away in importance as x approaches infinity.
7. C –substitution and properties of integrals
8. D –Product Rule and Chain Rule
9. A –First find the slope when $x = \pi$. The slope is positive 2. Use $(\pi, 0)$ in point-slope equation or slope-intercept equation.